

## Research Report

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# Expanding the *CBAL*™ Mathematics Assessments to Elementary Grades: The Development of a Competency Model and a Rational Number Learning Progression

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## RESEARCH REPORT

# Expanding the *CBAL*<sup>™</sup> Mathematics Assessments to Elementary Grades: The Development of a Competency Model and a Rational Number Learning Progression

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Prior work on the *CBAL*<sup>™</sup> mathematics competency model resulted in an initial competency model for middle school grades with several learning progressions (LPs) that elaborate central ideas in the competency model and provide a basis for connecting summative and formative assessment. In the current project, we created a competency model for Grades 3–5 that is based on both the middle school competency model and the Common Core State Standards (CCSS). We also developed an LP for rational numbers based on an extensive literature review, consultations with members of the *CBAL* mathematics team and other related research staff at Educational Testing Service, input from an advisory panel of external experts in mathematics education and cognitive psychology, and the use of small-scale cognitive interviews with students and teachers. Elementary mathematical understanding, specifically that of rational numbers, is viewed as fundamental and critical to developing future knowledge and skill in middle and high school mathematics and therefore essential for success in the 21st century world. The competency model and the rational number LP serve as the conceptual basis for developing and connecting summative and formative assessment as well as professional support materials for Grades 3–5. We report here on the development process of these models and future implications for task development.

**Keywords** Rational number; learning progression; competency model; mathematics

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## Introduction and Rationale

Prior work on the *CBAL*<sup>™</sup> mathematics competency model (Graf, 2009; Graf, Harris, Marquez, Fife, & Redman, 2009; Haberstroh, Harris, Bauer, Marquez, & Graf, 2010) resulted in an initial model for middle school grades that reflects current understanding of core skills in mathematics and their connections (e.g., as exemplified by the National Council of Teachers of Mathematics focal points). The *CBAL* mathematics competency model addresses both *content* and *process* strands for Grades 6–8. Cross-cutting processes include model, represent, and argue. Content-specific procedures and language include topics involving algebra, numbers and operations, measurement and geometry, and data analysis and probability. More general competencies are specified with respect to their more specific components or subcompetencies. Later work (Haberstroh et al., 2010) expanded and revised this model and defined cognitively based learning progressions (LPs) in order to support the development of assessments that would provide the evidence needed to enhance classroom use.

In the current project, starting from the existing middle school model, we developed a new *CBAL* mathematics competency model for Grades 3–5. The *CBAL* middle school mathematics competency model was originally developed 3 years prior to the release of the Common Core State Standards (CCSS), and thus we were striving not only to have the new elementary mathematics competency model be more suitable for younger students, but also to have a closer alignment with the CCSS so that it may be of greater use in today's classrooms. In particular, we made deliberate attempts to use the CCSS language and terms to ease the understanding of the model by school teachers and administrators and avoid confusion as much as possible. We thus began this project with a complete review of the CCSS for mathematics in Grades 2–5 with specific attention both to content standards and practice standards. We also reviewed the sixth grade standards as the linking piece between elementary and middle school mathematics. We elaborate on the process and reasons for the

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development of the new competency model for Grades 3–5 in the first section of this report. The *content* strands of the elementary mathematics competency model are derived largely from the CCSS with an influence from the middle school competency model. The determination of the *cross-cutting processes* appropriate for elementary school is more a result of examining both the CCSS and the strands of proficiency of Kilpatrick, Swafford, and Findell (2001) that served as the basis for the previously developed middle school competency model.

In addition to the new elementary school competency model, we also developed a new LP for rational numbers. The development of this new LP expanded on the process used for previous work where LPs were developed as a basis for assessment design (Arieli-Attali, Wylie, & Bauer, 2012), adding student and teacher interviews and cognitive laboratories to the literature synthesis and advisory panel input. This process began with an extensive literature review, aiming to identify lines of research and main findings and to pinpoint central big ideas or cognitive concepts that have the explanatory power of developing understandings. Following this review, a draft of the LP was developed and cognitive interview tasks were created to inform the LP. An advisory panel of experts then reviewed the LP and the cognitive interview results and suggested revisions. Out of this review, the LP presented here was modified to receive panel endorsement.

## Why Learning Progressions?

By articulating a trajectory of learning and understanding in a domain, LPs can provide the big picture of what is to be learned, support instructional planning, and act as a guide for formative assessment (Heritage, 2008). There is evidence that superior teachers use a conceptual structure similar to an LP (Clements & Sarama, 2004b). For example, in one study of a reform-based curriculum, the teachers who had the most valuable in-class discussions saw themselves not as moving through a curriculum but as helping students move through a progression or range of solution methods (Fuson, Carroll, & Drucek, 2000); that is, they were simultaneously using and modifying a type of learning trajectory (Clements & Sarama, 2004b). Simon (1995) discussed the knowledge of a *hypothetical learning trajectory* (HLT) as being essential to developing pedagogical thinking. Simon elaborated on this notion in 2004, demonstrating how thinking about the learning process and engaging in reflective abstraction promotes student learning (Simon, 2004).

The documentation of LPs can also be useful in the creation of proper diagnostic tools for formative assessment. By mapping LPs and developing an assessment around them, the assessment system can provide teachers with information regarding the location of their students on the progression and from that derive information needed to move the students forward. The CBAL research initiative has been using LPs as the theoretical basis for its formative and summative assessments. The CBAL definition of an LP from Educational Testing Service (2012) is as follows:

In CBAL, a learning progression is defined as a description of qualitative change in a student's level of sophistication for a key concept, process, strategy, practice, or habit of mind. Change in student standing on such a progression may be due to a variety of factors, including maturation and instruction. Each progression is presumed to be modal—i.e., to hold for most, but not all, students. Finally, it is provisional, subject to empirical verification and theoretical challenge. (para 1)

Other definitions for LPs exist. According to Confrey and Maloney (2010), a learning trajectory is “a researcher-conjectured, empirically supported description of the ordered network of constructs a student encounters through instruction . . . in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, toward increasingly complex concepts over time” (p. 968). Confrey and Maloney mapped several learning trajectories, one of which is directly relevant for our study, as it describes a progression for equipartitioning (as basis for fraction understanding). This progression names 16 proficiency levels for Grades K–7 and takes the shape of a *matrix* where the 16 proficiency levels are mapped in conjunction with 13 *task classes*. This matrix suggests another way to define an LP, and thus we took a careful look at these proficiency levels and this design as initial inspiration toward the development of our progression.

In 2004, *Mathematical Thinking and Learning* devoted a special issue to the topic of learning trajectories in mathematics (Clements & Sarama, 2004a). Additionally, the Rational Number Project (cf. Post, Cramer, Harel, Kieren, & Lesh, 1998) has 30 years of research by well-respected mathematics education researchers who often point to models for mathematics learning. Overall, models such as these describe the conceptual changes students are likely to make as they move toward proficiency in a domain. Some other example models are described in Steffe (2004) for fractions; Jones, Langrall, Thornton, and Mogill (1997) for reasoning about probability; Weaver and Junker (2004) and Doubler et al. (2011) in proportional

reasoning; Kalchman, Moss, and Case (2001) for rational numbers and functions; Carraher, Smith, Wiser, Schliemann, and Cayton-Hodges (2009) in measurement; and Koedinger, Alibali, and Nathan (1999) in algebra.

Beyond this work, there is a wide body of work on learning trajectories and progressions and their use in enhancing teaching and instruction. Literature on LPs and their use for assessment design in CBAL can be found in Harris, Bauer, and Redman (2008).

## Why Rational Numbers?

The Rational Number Project at the University of Minnesota (see Post et al., 1998) named fraction and decimal as topics that lie at the heart of rational number reasoning and, therefore, the heart of elementary mathematics. Analyses of the components of the concept of rational number suggest that this concept connected to most other topics in mathematics school learning. R. S. Siegler et al. (2012) have found that elementary school students' knowledge of fractions and of division uniquely predicts those students' knowledge of algebra and overall mathematics achievement in high school. This prediction stands even after statistically controlling for other types of mathematical knowledge, general intellectual ability, working memory, and family income and education. Thus, the CBAL project chose rational numbers as the priority LP in the initial creation of formative and summative assessments for Grades 3–5.

## Evidence-Centered Design Approach

The approach we are using in this study is the evidence-centered design (ECD) approach developed by Mislevy and colleagues (e.g., Mislevy, Steinberg, & Almond, 2003) as consistent with previous work on CBAL. The ECD approach calls for a *domain analysis stage* as a first step to building an assessment. Domain analysis includes the background information needed for the development of a *conceptual assessment framework*. Our domain analysis started with a review of the relevant literature in cognitive science and mathematics education to extract the main findings relevant to early mathematics concepts with specific focus on difficulties, misconceptions, barriers to learning, and aspects or principles of cognitive development relevant to the acquisition of early mathematics concepts. We pinpoint central *big ideas* or cognitive concepts in order for them to serve as the building blocks of the student model.

We continue with the second component of building the *student model* (i.e., the competency model of mathematics skills and knowledge for Grades 3–5 and relevant LPs). This building included a synthesis of the literature as well as taking into account the main topics in the CCSS. Any developmental sequence we may adopt has to take into account the teaching sequence that takes place in schools for at least these two reasons: (a) developmental stages of understanding are not independent of curriculum, and (b) we want our conceptual model to be applicable in the current school context in the United States. We checked and refined our LP model based on evidence collected from cognitive interviews conducted with students of Grades 3–5 and elementary teachers who teach mathematics. It has been largely recognized that cognitive interviews and think-aloud methodology can serve as good sources of information about student cognitive processes in problem solving and mathematics (Ericsson & Simon, 1993; Ginsburg, 1997; Newell & Simon, 1972) and specifically in the service of developing cognitive models of task performance for a cognitive diagnostic assessment (Leighton, 2004; Leighton & Gierl, 2007). Leighton argued that “relying on adult inferences [of students' knowledge] is a risky endeavor without independent empirical tests . . . from students who actually respond to the items” (p. 8).

While developing an LP for rational numbers, we explored the possibility of developing two distinct trajectories (i.e., one for fractions and one for decimals) but concluded that one unifying progression is needed. This was a result of our focus on detecting a concept-based developmental sequence that emphasizes *conceptual understandings* before other aspects (e.g., procedural knowledge, representational fluency). We made an explicit attempt to define clear distinctions between levels in order to enable clear interpretation of evidence when assessing those levels.

The next step is to develop an *evidence model*. An evidence model determines what types of student responses are relevant to the construct/competency measured and how these responses serve as evidence supporting the claim about student level of competency. To that end, we developed task models in which we determine relevant components that should be included in the task, some of which can be kept constant and some of which can vary. Each task model includes mathematical characteristics that are linked to the competency model and the LP (which together constitute the student

model) and an illustration of a scenario-based task that could exemplify one aspect of the model (see Appendix A for one sample task model). Following this approach, our research goals are described as follows:

1. The middle school competency model was developed prior to the CCSS. Our first goal was to create a new model that is relevant for elementary school, based on the existing model, and consistent with the CCSS.
2. Our next goal was to develop a provisional LP model for rational numbers (i.e., increasing sophistication in understanding of specific concepts and the relations between them) based on sound research from the learning and cognitive sciences. We chose rational numbers as this concept is a central topic in elementary school mathematics with implications for future mathematical understanding through algebra and beyond.
3. We aimed to conduct cognitive interviews to inform and refine the LP. We wished to discover which evidence could be collected to reflect different levels in the progression and provide a basis for the development of task models.

### Cognitive Interviews

Fourteen students in Grades 3, 4, and 5 (five students, four students, and five students, respectively; of those students, nine were female and five were male) were led through semistructured cognitive interviews with sample tasks designed to elicit evidence of understanding on the basis of the draft LP. We also interviewed two elementary mathematics teachers with the same tasks, asking them both to solve the problems and to simulate a hypothetical student responding to the same tasks. The students' cognitive interviews were used to test our hypothetical LP, and slight changes were made to both the progression and the tasks as we continued to conduct the interviews. Each participant (student or teacher) was invited to one individual 40–60-minute session, and the session was videotaped. (See the section “Evidence from Cognitive Interviews” for sample tasks.)

### Advisory Panel

We hosted a 2-day meeting of experts in the relevant domains of mathematics education and cognitive psychology research to present the rational number LP and to discuss potential task models.<sup>1</sup>

The panel read a draft version of the rational number LP as well as a thorough review of the supporting literature that we provided. During the meeting, they also viewed videos of portions of cognitive interviews and discussed new types of tasks that could be used in the assessment of the progression. Overall, there was strong support for the hypothetical progression, with clarifications and descriptions added as a result of the discussion.

## Competency Model for Elementary Mathematics

The existing CBAL middle school mathematics competency model addresses both *content* and *process* strands for Grades 6–8. Cross-cutting processes originally included model/represent and argue/justify. Content-specific procedures and language include topics involving algebra, numbers and operations, measurement and geometry, and data analysis and probability. More general competencies are specified with respect to their more specific components, or subcompetencies. Later work (Haberstroh et al., 2010) expanded and revised this model and defined developmental levels in order to support the creation of assessments that would provide the evidence needed to enhance classroom use. In the revised model, the model and represent competencies were separated into two subbranches so that the new model included three cross-cutting main processes: model, represent, and argue. Additional cross-cutting subprocess competencies have been added to round out the competency model with respect to the types of thinking described in Kilpatrick et al. (2001), especially the types of thinking in the adaptive reasoning braid and, to a smaller degree, the strategic competence braid. These new competencies include justify, use qualitative reasoning, connect, generalize, and use metacognition. Later on, another main branch was added that includes the three dimensions of conceptual depth and breadth, procedural fluency, and representational fluency. Figure 1 illustrates the middle school competency model.

To begin the elementary school mathematics competency model, we began with the CBAL middle school mathematics competency model. We then analyzed the Standards for Mathematical Practice from the CCSS in





Figure 1 CBAL middle school mathematics competency model.

mathematics to review its relationship with the *cross-cutting processes* in the middle school model. Those standards are as follows:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning (National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010, para 2–9).

We noted a number of points:

- Most of the Standards of Mathematical Practice are covered by the middle school competency model's use of cross-cutting mathematical processes dimension. For example, the argue strand aligns well with the cross-cutting process labeled construct viable arguments and critique the reasoning of others.

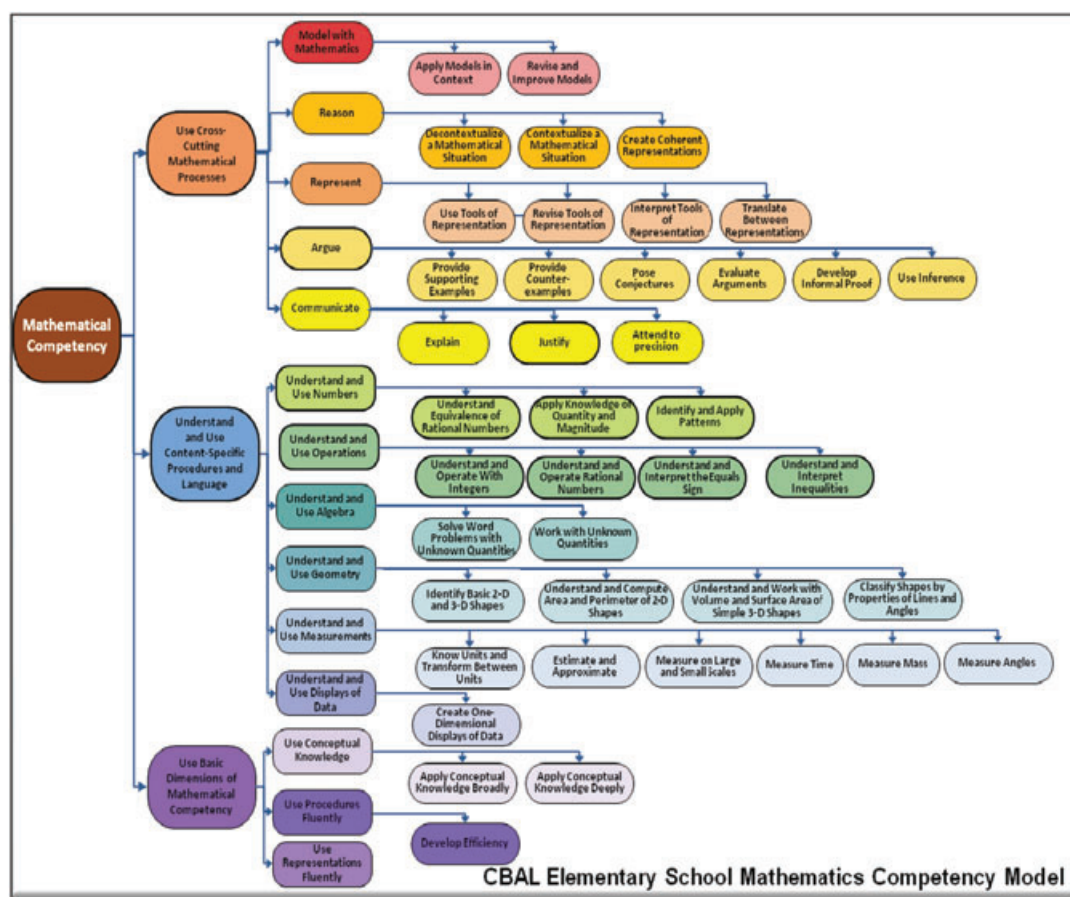


Figure 2 CBAL elementary school mathematics competency model.

- The use of the term *model* in the CCSS was different in meaning from the use of the same term in the middle school competency model. While the middle school competency model included both implicit and explicit models, the CCSS definition referred only to explicit models.
- Several concepts that were part of subnodes in the middle school model were featured more prominently in the CCSS. These are described below in the changes to the model.

We made the decision to use the language of the CCSS whenever terminology differed from the middle school model to make the competency model more accessible to teachers, who are working to become comfortable with the CCSS. We then formed our new elementary school competency model (Figure 2). This section highlights differences between this new model and the middle school model.

## Using Cross-Cutting Mathematical Processes

### *Model With Mathematics*

This node was renamed from the former model to make it clear that the definition of *model* is different from the term was used in the middle school model. In this case, we use the definition from the CCSS in which *model* refers to the application of mathematics to everyday problems. This use, therefore, only covers a subgroup of the former nodes associated with model: apply models in context and revise and improve models.

### *Reason*

This node was added to the elementary school model to capture an idea that may have been implicit in the middle school model but is explicit in the CCSS and that we felt was important to reflect. This strand now includes decontextualize a



mathematical situation, formerly referred to as abstract to models under the model strand in the middle school model. Two new nodes were added: contextualize a mathematical situation and create coherent representations, both of which are highlighted in the CCSS.

### ***Represent***

The changes to this strand were made due to the target population of elementary school students. We removed create representational devices and replaced it with use representational devices. We also removed integrate representational devices.

### ***Argue***

We removed the node recognize need for formal proof due to this being unnecessary for elementary mathematical competency.

### ***Communicate***

This strand was formerly a node between represent and argue. We pulled it down into its own strand to make explicit the very important nature of communication in elementary mathematics (and, we believe, all mathematics). The ability to communicate a mathematical idea or thought is essential for the demonstration of mathematical competency. In this strand, we pulled down justify, which was previously in the same strand as communicate, and added explain and attend to precision, both essential parts of communication in CCSS. We removed use metacognition, use qualitative reasoning, connect, and generalize as they were all covered via other nodes in more specific terms.

## **Understand and Use Content-Specific Procedures and Language**

This section of the competency model bears less resemblance to its middle school equivalent than the preceding section. This is due to the nature of the mathematics at the elementary level. In some cases, a node from the middle school model was broken up into several different subcomponents that should each be treated as a different competency for elementary school students. In other cases, mathematical competencies that were beyond the scope of elementary school mathematics were left out of this competency model. Specific examples of this are below by strand.

### ***Understand and Use Numbers***

The middle school model strand of understand and use numbers and operations was broken into two strands for the elementary school model. This separation was done to highlight the specific competencies that elementary school students must achieve, which may deal specifically with number or operational understanding at this age. Within this strand, we further broke apart the previous nodes to become understand equivalence of rational numbers, apply knowledge of quantity and magnitude, and identify and apply patterns. The other previous nodes were included in the following strand, or were removed because they were beyond an elementary school understanding.

### ***Understand and Use Operations***

The second part of the middle school strand for understand and use numbers and operations focuses on the manipulation of numbers, as opposed to the understanding of the numbers themselves. Further, we broke the nodes into understand and operate with integers, understand and operate with rational numbers, understand and interpret the equals sign, and understand and interpret inequalities.

### ***Understand and Use Algebra***

This node has been reduced to solve word problems with unknown quantities and work with unknown quantities, the two major components of elementary school algebra.

### ***Understand and Use Geometry***

Again, we broke down a previous node (work in 2- and 3-dimensional space using distance and angle) into several main components: Identify basic 2-D and 3-D shapes, understand and compute the area and perimeter of 2-D shapes, understand and work with the volume and surface area of simple 3-D shapes, and classify shapes by properties of lines and angles.

### ***Understand and Use Measurement***

While measurement in the middle school model focused more on formulas and calculation of measurements, measurement at the elementary level focuses more on understanding different types of units of measure and how and when to use them. Thus, this strand now has six nodes: know units and transformation between units, estimate and approximate, measure on large and small scales, measure time, measure mass, and measure angles.

### ***Understand and Use Displays of Data***

This final content strand presents another sharp departure from the middle school model. At this grade level, students need not calculate probability and interpret descriptive statistics as in the middle school grades. This strand has one node: create 1-D displays of data.

### ***Use Basic Dimensions of Mathematical Competency***

This section of the competency model refers to the relationship between the other two strands and is content free. Thus, there were very few differences between the elementary school model and the middle school model. The only change was the addition of a subnode to use procedures fluently, which is develop efficiency. The other strands, use conceptual knowledge and use representations fluently, remained unchanged.

## **Provisional Learning Progression for Rational Numbers**

While our initial plan called for two separate learning trajectories (fractions and decimals), the relevant literature stressed that it is the connections between these skills that is most indicative of mathematical success, not just in the early grades but also into high school (see Confrey, 1994, and also Moss & Case, 1999; Resnick et al., 1989; Thompson & Saldanha, 2003). For this reason, we decided on a structure that allows these learning trajectories to connect to each other to create one larger map that is focused on the concepts behind the understanding of rational numbers and the corresponding representations. When presented to our advisory panel, there was unanimous agreement that this decision was legitimate and wise and that our mappings between the two systems were clear.

We will first draw upon the literature review we have conducted and the big ideas we identified, which helped and created the backbone of the levels of the progression.

### **Literature Review: The Emerging Understanding of Fractions and Decimals**

In the fractions literature, we identified several main approaches to study the understanding as well as the teaching of fractions: (a) through understanding of fraction as representing several different types of entities, such as part/whole, part/group, point on the number line, ratio, quotient, operator (e.g., Behr, Lesh, Post, & Silver, 1983; Kieren, 1976; Novillis, 1976); (b) through the types of representations one can use to explain fractions (see Andrade, 2011; Common Core Standards Writing Team, 2011); and (c) through the ability to explain fractions and operations on fractions in context (see Ma, 1999).

#### ***Fraction as Representing Several Different Types of Entities***

One of the earliest works is by Novillis (1976), who created a hierarchical structure depicting the foundations of fractional understanding. The Novillis model, titled a hierarchy of selected subconcepts of the fraction concept (HSSFC), associates

various concepts of fractions with their related models. The hierarchy was tested with fourth through sixth graders on the basis of a fraction concept test created by the author. Connections were then confirmed or disconfirmed on the basis of whether the empirical results violated or confirmed the hierarchical predictions. From these results, the author identified several prerequisites for associating a fraction with a point on the number line, including associating fractions with *part-whole model* and *part/group model*. The author also noted that many students can associate the fraction  $1/5$  with a set of five objects, one of which is shaded, but most cannot associate  $1/5$  with a set of 10 objects, two of which are shaded. It is worth noting that in the literature of recent decades, the part/whole model and the part/group model are typically no longer treated as a different constructs and both are generally referred to as part/whole.

Kieren (1976) was the first to propose that the concept of fractions consists of several subconstructs and that understanding the general concept depends on gaining an understanding of each of these different meanings of fractions as well as of their connections. Kieren (1976) initially identified four subconstructs of fractions: *measure*, *ratio*, *quotient*, and *operator*. In the author's original conceptualization, the notion of the part-whole relationship was considered the seedbed for the development of the other subconstructs; thereby, Kieren (1976) avoided identifying this concept as a separate, fifth, subconstruct claiming that this notion is embedded within all other subconstructs. In the following years, Behr et al. (1983) further developed Kieren's (1976) ideas recommending that the *part-whole* relationship comprise a distinct subconstruct of fractions. They also connected this subconstruct with the process of partitioning. Several researchers emphasized the need for children to build a deep understanding of fractions by using a variety of concrete and pictorial models and partitioning activities (e.g., Hunting, 1983; Kieren, 1976; Kieren, Nelson, & Smith, 1985; Piaget, Inhelder, & Szeminska, 1960; Pothier & Sawada, 1983, 1984), and indeed, instructional programs endorsed this perspective. However, not much has been done in the instructional programs to ensure the shift from a strong part-whole representation to an understanding of a fraction as a single number, and research has shown that the essential mistake of many elementary school students is to interpret fractions as *pairs of whole numbers* (Smith, 2002). For example, Smith (2002) found that when he asked his students whether there are fractions between  $3/5$  and  $5/7$ , one of the students answered:  $3/6$ ,  $3/7$ , . . . ,  $3/12$ ;  $4/2$ ,  $4/3$ ,  $4/4$ , . . . ,  $4/12$ ; and  $5/2$ ,  $5/3$  . . .  $5/6$ . In particular, researchers have argued that for future understanding of relations and operations it is critical to make the linkage of numbers (fractions) with their *referents*, that is, the linkage of units of measure (the notation of a fraction) with their *magnitude* of quantities (the single number represented by this notation; Behr, Harel, Post, & Lesh, 1993; Harel & Confrey, 1994; Hiebert & Behr, 1988; Kaput, 1985). Lamon (2001) argued that a *number sense* needs to be acquired to allow anticipation of the outcome of the equipartitioning procedure (what each person should get in an equipartitioning scenario), and this number sense will progress from a basic level of understanding to higher levels of understanding. Thus, reflected in the literature is the dual-perspective of a fraction as part-whole concept and as a single number concept (a measure) and the importance of linking or shifting from the basic notion of the former to the more advanced understanding of the latter, which in turn will enable better future understanding of operations with rational numbers.

One specific property of fractions as *quotient* received special attention: Graeber and Tirosh (1990); Lamon (1999, 2001, 2007), and Toluk (1999, as cited in Oksuz & Middleton, 2007) all showed that the understanding of  $a/b$  as indicating a quotient is rare among students. Lamon (2001) further noted that poor understanding of  $a/b$  as indicating division can lead to many problems in high school calculus. Toluk (1999, as cited in Oksuz & Middleton, 2007) studied four children in a series of parallel individual teaching experiments. The author found that children progress from seeing whole-number division and fair sharing as two different domains (division and fractions) to seeing fractions in terms of division. In the analysis, the author came up with a model describing the students' development: First, the children had wholly separate conceptions of fractions-as-part/whole and division-as-whole-number quotients, then when students were asked to subdivide a remainder in fair-sharing contexts, they eventually came to see the possibilities of fractional quotients describing cases where the numerator is larger than the denominator. Finally, over time, the situational and notational analogies presented for fractions-as-fair-share and division-as-fractions allowed students to conceive of both fractions and division as being the same thing. However, it should be noted that Toluk (1999, as cited in Oksuz & Middleton, 2007) instructed these children toward this progression and this realization very rarely occurs naturally with students. In fact, Ma (1999) showed that many teachers in the United States are unable to construct the situational analogies necessary for this transformation.

The understanding of a fraction as a *ratio* is most critical to subsequent understanding of proportional reasoning. One of the common misconceptions among students is to think that a ratio of  $3/5$  is equivalent to a ratio of  $5/7$ , because the

difference between 3 and 5 is 2, which is the same as that between 5 and 7. According to Smith (2002), this misconception is a result of the essential mistake in interpreting fractions as pairs of whole numbers (that is, in this case, viewing  $\frac{3}{5}$  as 3 and 5). (See the section titled “Challenges in the Acquisition of Rational Number Concepts,” later in this report for more on this and other similar errors.)

### ***Decimals as Primarily a Notation System and Part of Base-10 Understanding***

The decimal literature is not as rich and long-standing as is the study of fraction. In the decimal literature, we identify two main approaches to study students’ difficulties: one that focuses on the misconceptions of decimal notations and the other that studies decimal numbers as part of the wider rational number system (i.e., in relation to fractions).

Studies that focus on decimal notation identify notation errors as indications of conceptual difficulties originated from either prior knowledge of whole numbers or from erroneous incorporation of general ideas of fractions (e.g., Nesher & Peled, 1986; Resnick et al., 1989; Roche, 2005, 2010; Sackur-Grisvard & Leonard, 1985; Stacey & Steinle, 1999; Steinle & Stacey, 2004). Two kinds of error are observed when students are asked to compare decimal numbers: (a) *longer is larger*, in which a student may decide that 0.125 is larger than 0.3 because the longer the decimal portion, the larger the number, or alternatively asserting that 125 is larger than 3 (that is, the student reads the numbers after the decimal point as whole numbers) and (b) *shorter is larger* in which a student may decide that 0.3 is larger than 0.35, because in the first number the whole is divided into tens and in the later the whole is divided into hundreds. Research had found both of these errors to be repeated over time by students, though persistence is more prevalent for the first kind (assumed to be originated from whole number understanding) and less frequent for the second kind (that may be derived from fraction understanding). Some studies have elaborated on these two kinds of errors, producing additional subcategories and more refined classifications (e.g., Steinle & Stacey, 2004). For example, Sackur-Grisvard and Leonard (1985) suggested steps in the acquisition of comparison rules for decimal numbers in which children are assumed to acquire first understanding of the presence of the point, followed by understanding of the property of the place value, and finally understanding of the property of zero.

The decimal number system is first and foremost part of the base-10 understanding and place-value representation. According to Sinclair and Scheuer (1993), understanding of written numerical notations is a construction process that is necessary to the understanding of our numeration system, and it participates in and directly influences mathematical cognition. Studies investigating place-value understanding with whole numbers show that young children encounter much difficulty with these concepts. In the Sinclair and Scheuer study with first-grade children, for example, although all children correctly identified that numbers with more digits are greater than numbers with fewer digits (when only positive integers were used), many of their comparisons of numbers having the same number of digits were wrong, such as saying that 19 is greater than 21 because one and nine is bigger than two and one. Another interesting finding relating to the difficulty in grasping the place-value concept was that more than half of the children interpreted digits as corresponding to their *face value*; that is, while recognizing that the entire number together stood for the entire number of tokens, the digits when taken separately each stood for the number of tokens of the single digit quantity (such as explaining that the digits 1 and 5 in the number 15 stood for one token and five tokens, respectively). Some of the children knew that this explanation was not adequate and that somehow all of the tokens had to be accounted for, yet they did not understand how—even adding extra digits for the leftover tokens despite previously stating that the original number stood for all of the tokens (e.g., adding the digit 9 onto the end of the number 15 in the previous example to account for the nine tokens leftover). In a related study, in individual interviews with children from widely diverse urban, rural, public, and private school communities, Ross (1986) presented children with 25 sticks and asked them to count the sticks and write down the number. After this presentation, Ross circled the number 5 and asked, “Does this part have anything to do with how many sticks you have?” then subsequently indicated the number 2 and asked the same question. The children’s responses were categorized into four levels: in Level 1, the individual digits have no numerical meaning; in Level 2, the child invents a meaning for the individual digits unrelated to place value; in Level 3 the individual digits have some meaning related to place value, but it is a partial and confused idea, such as the ones place indicating tens or both places indicating ones; and in Level 4 the digits represent the whole quantity partitioned into groups of tens and ones. Ross found that it was not until Grade 4 that half of the children in the class reached Level 4. By Grade 5, only two thirds of the children knew that the 5 meant five sticks and the 2 meant

twenty sticks. This finding is surprising as children of this age are taught arithmetic algorithms based on an understanding of place value such as long division, multidigit multiplication, subtraction with borrowing, and addition with carrying. Additionally, these results are not much better than Sinclair and Scheuer's (1993) results derived from first graders.

Silvern and Kamii (1988, as cited in Kamii, 1989) carried out an almost identical study using 16 tokens with very similar results, except that this time only 35% of fourth graders gave a mathematically sound answer. One possible explanation for this discrepancy is that children at this age are still using number words to guide their notational understanding, so it may be easier to extrapolate that the digit 2 in the number 25 is 20 than to realize that the digit 1 in the number 16 is 10. In fact, teen numbers have been found to be harder for children to understand in terms of the relationship between written and spoken numbers (DeLoache & Willmes, 2000). Ross (1986); Sinclair and Scheuer (1993), and Silvern and Kamii (1988, as cited in Kamii, 1989) all showed that children in the elementary grades (one through five) are still having difficulties in the interpretation of written numbers and of place value even with whole numbers.

One conclusion from studies that concern decimal-notation difficulties is the need to connect the decimal concept and the fraction concept with the meaning of a number. One suggestion is to use fractional language in the teaching of decimals (Roche, 2010), for example, to speak of 2.75 as 2 and 75 hundredths (instead of two point seventy five) and to avoid rules that encourage whole-number thinking. Roche (2010) further suggested using different representations to elicit understanding of the fractional aspects of decimals, similar to the ones that are used for teaching fractions. A few examples of suggested representations are linear arithmetic blocks (Helme & Stacey, 2000), Deciwire (a decimal number line; Young-Loveridge & Mills, 2011), and Decimat (an area carpet partitioned to decimals; Roche, 2010).

In contrast to the above representational approaches, Moss and Case (1999) argued that “the order of teaching fraction-decimal-percent is more *arbitrary* and that what matters is that the general sequence of coordinations remains progressive and closely in tune with children's original understanding” (p. 125, emphasis added). Moss and Case (1999) went on to suggest a curriculum where the order of teaching is first percent, followed by decimals, and ending with fractions. In an experiment, they found that the students who received the experimental curriculum showed a deeper understanding of rational numbers than those in the control group, as well as reliance less on whole-number strategies when solving novel problems. In a different study Resnick et al. (1989) found similar results when comparing students from the United States, Israel, and France, where each country had a different curriculum. Specifically, the French curriculum included introducing decimals prior to fractions. Their conclusion was that misconceptions or difficulties with fractions or decimals are related to prior knowledge and the order in which these concepts are taught.

## Challenges in the Acquisition of Rational Number Concepts

Given the vast number of studies concerning rational-number difficulties, it is clear that researchers acknowledge the challenges in the learning of these concepts. Moss (2005) summarizes several challenges derived from research findings. The primary challenge the author notes is the necessity to move from absolute to relative thinking (i.e., one must always keep in mind what the “whole” is), along with the acquisition of a new symbol system. Rational numbers are also the first time that elementary students see a divergence from a one-to-one correspondence of symbol to referent. That is, there are different representations of the same quantity (decimal, fraction, percent) and infinite equivalent fractions and different meanings (subconstructs) of the same fraction (as we talked about in length above; cf. Behr et al., 1983; Behr et al., 1993; Behr & Post, 1992; Behr, Wachsmuth, Post, & Lesh, 1984; Kieren, 1995; Ohlsson, 1988). For example, if previously a student knew that the number 2 stands for a group of two objects, now the symbol  $\frac{2}{5}$  is (a) part/whole (i.e., 2 of 5); (b) division (i.e., 2 items divided between 5 people); (c) ratio (i.e., 2– 5 ratio); (d) a measure (i.e., 0.4; fixed quantity, number line representation); and (e) multiplicative operator (i.e., operator that reduces  $\frac{2}{5}$  or enlarges  $\frac{5}{2}$  the size of another quantity). Also, translation between representations does not come easily (Markovits & Sowder, 1991, 1994; Moss, 2005; Sowder, 1995). For example, Moss found that more than half of sixth and eighth graders asserted that  $\frac{1}{8}$  should be represented as 0.8 (rather than the correct answer 0.125). Ordering rational numbers that are presented in mixed representation (some as fractions and some as decimals) turns out to be a difficult task (Moss, 2005; Sowder, 1995).



Further difficult concepts need to be acquired as well. For instance, a fraction needs to be perceived as two numbers for some of the representations, and as one number for others. There are also previously ingrained ideas that interfere with the new rational number concepts. For instance, numbers are now *dense* (between any two numbers, there are other numbers). This understanding also acts as a precursor for continuous versus discrete concepts in later mathematics. Additionally, students must deal with a new method of ordering numbers: the smaller the denominator the bigger the fraction given the same numerator. Decimal notation also reverses previous facts about place value: if you add zero to the end, the number stays the same.

The relationship between numbers is also different than whole-number arithmetic: for one, numbers have multiplicative relationships, rather than additive. For example, students must grapple with how it is that two whole numbers (say 2 and 50) exist in a relationship that creates a new number whose value is smaller than 1. Similarly, the unit is implied but not explicit ( $1/2$  of  $1/8$  is answered by 75% of fourth and sixth graders as  $1/4$ , mistaking the unit to be 8 and not 1; Moss, 2005).

As shown previously, rational number understanding involves a wide range of concepts relating to fractions, and additional concepts relating to decimals. Table 1 shows a summary of the predominant concepts, or the big ideas, as they appear in the literature. Listing the big ideas helps in the building of the levels of the progression.

### Rational Number in the Common Core

While the CBAL project began 3 years prior to the release of the CCSS for mathematics, we are striving for all new LPs to have a strong level of agreement with the CCSS so that they may be of use in today's classrooms. We thus began with a complete review of the CCSS for mathematics in Grades 2–5 with specific attention to the standards involving fractions and decimals. We also reviewed the sixth grade standards that mentioned ratios and percents as we see these representations as exceedingly important toward the conceptual understanding of rational number, but they are not included in the common core until Grade 6.

This review found that the teaching of fractions according to the CCSS starts in second grade with the *part-whole notion* and *partitioning procedures*, mainly with two, three, or four parts of circular or rectangular areas. It further continues in third grade to the concept of *fractions as numbers* and the representation of *fractions on the number line* with attention to equivalent fractions. In fourth grade, the CCSS includes the teaching of operations with fractions, mainly addition and subtraction of fractions with common denominators, further developed in fifth grade to all four operations (addition, subtraction, multiplication, and division), including finding a common denominator when needed. This stage (fifth grade) is also when the notion of a *fraction as division* is introduced. Decimals in the CCSS take a different path. The teaching of decimals is part of the numbers and operations in base-10 domain, and decimals are introduced only in fourth grade. The CCSS does emphasize connecting the decimals to fractions (viewing decimals as a special case of fractions with denominators of 10, 100, etc.), but decimals do not get the same attention using partitioning procedures as the topic of fractions does. Specific attention is given to the notation of decimals, their equivalence to fractional notation (e.g., teaching techniques to convert fractions to decimal and vice versa), and decomposing decimals to other equivalent forms (e.g., expanded form, which emphasizes place-value structure, as in the case where writing a number like 32.45 as equivalent to  $3 * 10 + 2 * 1 + 4 * 0.1 + 5 * 0.01$ ). In fifth grade after the notation of decimals is established, operations with decimals are taught. There is no mention of connections or links between teaching operations with decimals and teaching operations with fractions.

Two major conclusions were drawn from the review of the CCSS:

1. The CCSS provides a teaching sequence that determines a specific order in which the part-whole concrete concept of fractions precedes the concept of a fraction as a number. This sequence is supported by cognitive development research, as the following review of the literature shows. However, the teaching of decimal numbers is primarily used as a notation that completes the sequence in the sense that it is placed in the later years of elementary school but without enough emphasis on the connection between fractions and decimals. This teaching sequence makes a distinction between the teaching of fractions and teaching of decimals as different units to be taught at different times and also makes explicit that each topic displays a different progression within itself.
2. Fractions precede decimals and percent in the CCSS. As the preceding review of literature showed, this sequence may not necessarily be the optimal one with regard to cognitive development. Nonetheless, the fact that this is the

Table 1 Big Ideas Students Need to Acquire in Relation to Rational Number Understanding

Big idea	Importance	Research emphasis
A fraction as part – whole and part/group representation	Pictorial representation (e.g., shaded parts of a partitioned pizza or shaded balls in a group of balls) is used with preschool and early grade school children and found to be a precursor for all other fractional concepts (Kieren, 1976; Novillis, 1976).	Partitioning, as a process rather than a result, is a backbone in several studies concerning fractions (e.g., Confrey, Maloney, Nguyen, Mojica, & Myers, 2009). Kieren (1995) and Empson and Turner (2006) used <i>folding tasks</i> to study and describe students' emerging understanding of fractional concepts and multiplicative thinking.
The concept of half and the halving procedure	The notion of halving is recognized as the fundamental procedure and a first step in any equipartitioning procedure (Confrey, 1995; Confrey et al., 2009).	A progression from <i>halving</i> (a partition into two parts) into equipartitioning to any number of parts may go through <i>repeated halving</i> (Davis & Hunting, 1991; Empson & Turner, 2006). Callingham and Watson (2004) found that uses of half, both as number ( $1/2 + 1/2$ ) and operator ( $1/2$ of ...) were the easiest items for mental computation.
The concept of a unit fraction	The development of a concept of a unit fraction that can be iterated to build other common fractions (e.g., $4/6$ as iterating 4 times the unit fraction $1/6$ ), but not beyond the original whole, has been identified as an indicator of progress in the development of fractional concepts still in the part – whole understanding of fraction (Hackenberg, 2007; Olive & Steffe, 2002; Steffe, 2004).	Improper fractions (e.g., seven times the unit $1/6$ that would yield $7/6$ ) do not make sense to a student who has not progressed beyond the understanding of the unit fraction, because it does not make any sense that the fraction is bigger than its whole (Steffe, 2002).
The conceptual understanding of improper fractions	The emergent understanding of improper fractions (Hackenberg, 2007; Steffe, 2004; Tzur, 1999) suggests that this further development requires a unit fraction conception and the iterative partitive scheme (cf. Steffe, 2004).	Iteration (i.e., repeatedly instantiating an amount in order to produce another amount, as demonstrated by Tzur, 1999) can help with creating and understanding an improper fraction. When iterating a proper fraction like $6/11$ two times (i.e., doubling $6/11$ ) children can produce and understand the improper fraction $12/11$ , because $12/11$ cannot gain meaning from being part of a whole.
A fraction as a measure and number line representation	A fraction perceived as a measure is one of the subconstructs identified by Kieren (1976). Understanding a fraction as a measure includes conceiving a fraction as a single number on the number line, understanding concepts of ordering and equivalence as well as fractional notation for numbers larger than one whole (improper fractions and mixed numbers), and translating between among notation systems, fractions, and decimals. The teaching of the concept of a fraction as a number on the number line follows (in the CCSS) the teaching of a part/whole concept of fraction.	Recently, the U.S. Department of Education recommendations for developing effective fractions instruction (R. Siegler et al., 2010) include a recommendation based on research findings to use number lines as a central representational tool in teaching fraction concepts, along with measurement activities (recommendation 2, pp. 19 – 25).
		Novillis (1976) found that part/whole representation is a prerequisite for a number-line representation. A number-line representation has been found to be useful for the development of number sense and magnitude sense of whole numbers (Booth & Siegler, 2008; R. S. Siegler & Booth, 2004) as well as rational numbers (Schneider, Grabner, & Paetsch, 2009; Sprute & Temple, 2011; Weller, Arnon, & Dubinsky, 2009).

Table 1 Continued

Big idea	Importance	Research emphasis
Equivalent fractions and decimals, and ordering fractions and decimals	Early conception of equivalence of fractions and ordering of fractions develop through visual models or manipulatives in a part-whole context and as a specific concrete embodiment that needs to be further developed to become independent of that embodiment (Post, Wachsmuth, Lesh, & Behr, 1985).	Post et al. (1985) argued that children's understanding about ordering <i>whole</i> numbers may impede the development of understanding of ordering rational numbers, because (a) fraction size depends on the <i>relation</i> between two numbers (the numerator and the denominator); (b) there is an inverse relation between the number of parts the whole is partitioned into and the size of each part; (c) when fractions are from the same partitioned whole (have like denominator), the order of the numerators follows whole number ordering; but (d) when the denominators are different, the judgment about their sizes requires flexible work involving conversion to common units; and (e) the density of the rational numbers implies the counter-intuitive notion that there is no next fraction.
Decimal as a notation system	Students struggle with decimal number understanding, and studies show that although the main difficulty is derived from the notation system, the problem is more conceptual in nature (Resnick et al., 1989)	Two main kinds of error are observed when students are asked to compare decimal numbers: (a) longer is larger, in which deciding that 0.125 is larger than 0.3 because the longer the decimal portion the larger the number; and (b) shorter is larger, in which deciding that 0.3 is larger than 0.496, because in the first number the whole is divided into tens and in the later the whole is divided into thousands (Roche, 2005, 2010; Sackur-Grisvard & Leonard, 1985; Stacey & Steinle, 1999; Steinle & Stacey, 2004).
Place value notation	According to Sinclair and Scheuer (1993), understanding of written numerical notation is a construction process that is necessary to the understanding of our numeration system, and it participates in and directly influences mathematical cognition.	Sackur-Grisvard and Leonard (1985) suggested steps in the acquisition of comparison rules for decimal numbers in which children are assumed to acquire first understanding of the presence of the point, followed by understanding of the property of the place value, and finally understanding of the property of zero.
A fraction as an operator	Understanding of a fraction as an <i>operator</i> was proposed by Kieren (1976) as one of the subconstructs of fractional knowledge.	Studies elaborating on this concept are, for example, Behr et al. (1983, 1993).
A fraction as quotient	In viewing fractions as <i>quotient</i> , fractions are mapped onto the state that results from the action of partitioning a set into a given number of groups; the unit of reference for the fraction is a single item in the set (Empson, Junk, Dominguez, & Turner, 2005).	Charles and Nason (2000) defined <i>partitive quotient fraction construct</i> as a conceptual mapping in an equal sharing problem between the <i>dividend</i> and the <i>numerator</i> , and between the <i>divisor</i> and the <i>denominator</i> , thus understanding that if you divide three sandwiches (the dividend) among five children (the divisor), each child will get $3/5$ (three-fifths) of a sandwich.
A fraction as a ratio and rate	Understanding a fraction as a ratio and a fraction as a rate also exist in the literature, but these topics do not enter the CCSS until Grades 6 and 7 and thus we do not include them in our LP.	

order in which children are taught has an implication for concept development, which needs to be considered in designing a formative assessment system.

### ***A Concept-Based Approach to Developing Rational Number Progression***

From the review of the literature, it seems that although the conventional order of teaching is from fraction concepts to decimals, this order is not the only sensible possibility. In fact, numerous studies show that these concepts go hand in hand and separating them for the purpose of teaching may not be ideal (e.g., Confrey, 1994; Moss & Case, 1999; Resnick et al., 1989; Thompson & Saldanha, 2003). Our conclusion is that we need to address all important aspects of fractions and decimals in one comprehensive trajectory in order to provide a strong basis for cognitively based assessment.

This sentiment was backed by Callingham and Watson (2004), who aimed at finding a comprehensive trajectory, but only for mental calculation. In their study, Callingham and Watson used item analysis with Rasch modeling techniques (specifically considering difficulty of items) to create competency levels for mental computational tasks of fractions, decimals, and percent with children from Grades 3–10. This empirical work, although only for mental computation, provides basic understanding regarding the relative difficulty of subtopics within each topic (fractions, decimals, and percent) and the relations among them. Inspired by this approach, we further develop our provisional LP for rational numbers to include both fractions and decimals, as two manifestations of a core conceptual trajectory.

### **The Learning Progression Model**

The model of students' understanding of rational number has two central concepts: a shift from a part/whole representation into a single number understanding and an integration of decimal and fraction notations and representations. Both of these notions develop and deepen from Level 1, the most basic level of understanding addressed by the model, to Level 5, the most sophisticated level. We begin with a detailed description of each level in regard to what students understand, what students can do, and what they might have trouble doing.

We acknowledge that change in student standing relies on both maturation and instruction, and thus, it is possible for a student to show evidence of one level with fractions, for instance, and a lower level with decimals because he or she has not yet received decimal instruction. The notion, however, is that the cognitive underpinnings necessary to achieve standing on a level may be present, even if instruction has not yet made possible the ability to perform at that level with both fractions and decimals.

### ***Progress Variables***

Progress variables are dimensions of knowledge that are used as a way of charting growth (see Corcoran, Mosher, & Rogat, 2009). As a result of the advisory panel discussions, we defined five progress variables to be used in this LP: fractional units, measure/fraction as number, additive structure, multiplicative structure, and strategic thinking/flexibility. These variables are defined and discussed next.

#### ***Fractional Units***

This progress variable refers to how the student perceives the relationship between quantities, between the part and the whole, and between the unit of the partitioned whole and a quantity that consists of the combination of several such units (see Steffe, 2004).

The fractional units perception develops from the basic concept of a half and a halving procedure. Young children have an understanding of sharing, where each one gets a part, but sometimes one can get the bigger half or the smaller half. This kind of understanding implies that the half is not an exact amount but rather a partitioning of an item or quantity into two parts. In that sense, there are no fractional units yet perceived, but these are the grounds on which they will develop.

The progress variable of fractional units develops in our progression through the first three levels (see Table 2). The first step in the development of fractional units is the perception of a quantity or a group as the *unit of one*

(the whole) accompanied by the perceiving of parts within wholes in a process of partitioning and acting on (Level 1 in our progression). The ability to perceive the parts separate from the whole, while capable of holding in imagination the image of the part as separate from the whole, is an advance in development (Level 2). At that stage, students have a notion of *units of units* (see Hackenberg, 2007; Steffe, 2004)—the unit fraction and the unit of the partitioned whole. The next level is the understanding that one can operate on the unit fraction independent of the whole and beyond the whole to create an improper fraction (Level 3) and thus have a conception of *units of units of units*: the whole, the fraction unit, and the improper fraction. In Levels 4 and 5, this notion goes through more fluency and flexibility and is in the background of other perceptions that are enabled due to the solid acquisition of fractional units understanding.

#### *Measure/Fraction as Number*

The first step in the development of the notion of a fraction as a measure is the ability to place a fraction on a number line of size  $[0,1]$ . In fact, this is not yet a measure understanding but rather a perception of partitioning the line  $[0,1]$  into  $n$  equal parts and finding the  $n$ th part according to the convention of starting to count from the left (the zero). In our progression, this development occurs in Level 2 along with the conception of a unit fraction discussed above. The next level of development (Level 3 in our progression) is conceiving of a fraction on a number line longer than 1 (e.g., placing a  $1/4$  on  $[0,2]$  or  $[0,5]$ , etc.) and of an improper fraction on that line (e.g., placing  $1\frac{1}{4}$  on  $[0,2]$  or  $[0,5]$ ). At that stage, specific common fractions (*benchmarks*) are easily identified on the number line (e.g.,  $1/2$ ,  $1/4$ ,  $3/4$ ) and other unit fractions, but only when the number line is partitioned to the number denoted by the denominator (e.g.,  $3/10$  can be placed on a number line that has  $1/10$  partitioning but not  $1/5$  partitioning). Only at Level 4 in our progression do children perceive of the fraction as a measure independently of the specific scale of the number line. The progress variable of a measure develops in our progressions in Levels 2–4.

#### *Additive Structure*

Following Vergnaud's (1994) research on additive structures, this progress variable refers to how students understand and apply the operations of addition and subtraction to fractions. Students begin at Level 1 with no means of applying addition and subtraction to fractions as fractions are based on repeated halving and not a unit of units (Level 2), as is necessary to begin adding and subtracting the fractions as units in their own right. However, in Level 2, students are not able to add and subtract fractions to sums greater than 1, as fractions are still sub-parts of a whole. By Level 3, when students are able to perceive of a fraction as a number, they are also able to add and subtract those numbers to achieve sums greater than 1. Students in Levels 4 and 5 are able to manipulate fractions additively with ease, just as they do whole numbers.

#### *Multiplicative Structure*

Similar to additive structures, this progress variable refers to how students understand and apply the domain of multiplication and division to fractions. Multiplicative structures are first apparent in Level 2, where students show early division concepts, such as being able to solve for one-half of one-quarter. However, this early understanding of fraction as operator is the only multiplicative structure available to the student until he or she recognizes fractions as numbers in Level 3, setting the path toward multiplicative multiplication and division, though these remain very difficult concepts until Levels 4 and 5.

#### *Strategic Thinking/Flexibility*

Students differ in their ability to apply different strategies at problem solving. During early stages of understanding, a student may have only one strategy available to solve a specific problem. When that student's understanding progresses, more strategies are at hand and the ability to choose the more efficient one, the one that will solve the problem quicker and easier, may be an indication of a higher level of understanding. Moreover, in real-world problems, part of the difficulty is sometimes to figure out exactly what the problem is and how to model it mathematically (this is related to the model and represent strands in our competency model). Kilpatrick et al. (2001) termed this ability *strategic competence*. Kilpatrick



Table 2 Progress Variable Development Through the Learning Progression

	Level 1	Level 2	Level 3	Level 4	Level 5
Fractional unit	Focus on one level of units (wholes)	Ability to keep in mind two units (the whole and the parts of the whole). This includes early anticipation/imagination of parts and is the start of symbolic representation.	Ability to keep in mind three units at the same time (the whole, its parts, and the new unit that is built from several parts, smaller or bigger than the initial whole; e.g., the whole is 2 apples, 1/4 of that amount is 1/2 of an apple, and 6/4 is equal to 3 apples). This also includes improper fractions.	Same as Level 3	Same as Level 3
Fraction as number	No concept of fraction as number	No concept of fraction as number	Ability to see a fraction as a single number and work with it as a measure in its own right	Fraction measure is independent of the scale on the number line	Same as Level 3
Additive structure	No additive structure	Beginning of additive structure	More advanced additive structure that allows for a sum greater than 1 and flexibility with the whole	More advanced understanding of the additive properties of fractions. Ability to manipulate fractions additively with ease.	Same as Level 4
Multiplicative Structure	No multiplicative structure	Early division concept, such as "What is 1/2 of 1/4?"	Ability to move beyond the early division concept to include fractions as numbers and not just operators	Ability to multiplicatively relate any two fractions. For instance, "transform 2/3 into 2/5 multiplicatively." Ability to apply the distributive property to fractions.	Advanced multiplicative concept, enabling the use of symbolic notation in a rich and meaningful way
Strategic thinking/ flexibility	Inability to work strategically (repeating of single method learned)	Start to see different options in equipartitioning. Using anticipation/imagination allows for early flexibility and strategic thinking. However, students may still depend on magnitude of numbers in using different options (that is, while with certain numbers like 1/2, 1/4 or even 1/8 they may see equivalent ways of partitioning, it may not be the case with 1/6 or 1/21).	Ability to exhibit some variety in the ways students solve problems due to their ability to refer to numbers as quantities. Being able to locate a fraction and a decimal on the number line opens up the possibility to see fractions and decimals as different representations of the same entity (the same quantity).	Advanced level of strategic thinking, whereby students have several different strategies to solve a problem and can choose the more efficient one for the specific problem. Ability to recognize the mathematical model within any problem. Ability to answer such questions as "How many 3/4s are in 2/5?" and other problems of rescaling.	Greatest amount of flexibility. Students have several different strategies to solve a problem, and they can choose the more efficient one for the specific problem. Ability to recognize the mathematical model within any problem. Ability to answer such questions as "How many 3/4s are in 2/5?" and other problems of rescaling.

et al. (2001) exemplified a common superficial method whereby students focus on the numbers in a problem and use so-called keywords to cue appropriate arithmetic operations (such as “less” to cue subtraction). However, a more proficient approach is to construct a problem model—a mental representation that maintains the structural relations among the variables in the problem. Referring to quantities instead of numbers, and the relations among them, is part of what a proficient student would do. Thus, some researchers refer to this ability as *relational thinking* (cf. Empson, Levi, & Carpenter, 2011).

See Table 2 for a summary of the development of these progress variables throughout the progression.

## Holistic Descriptions of the Learning Progression Levels

### Prior Knowledge (Level 0)—Half and Halving

Level 0 in our progression is in effect the prerequisite or the basic prior knowledge that students possess before entering elementary school and formal teaching. Based on the review of the literature (see Table 1) it is assumed that any student entering elementary school may already have a notion of halving from everyday experience that includes splitting into two parts.

### Level 1—Early Part/Whole Understanding

#### ***At This Level, Students Know . . .***

Students understand the relation between the parts that are *smaller than* and *embedded in* the whole. Students at this level also know and understand that the parts should be *equal*—though the meaning of *equal* may not be fully established, that is, some students may identify equal with congruent shapes. Although they understand equipartitioning to any number of parts, partitioning to even number of parts, and specifically to  $2^n$  number of parts, is easier and allows for repeated halving strategy.

At the same time, students may have basic understanding of money units and the part–whole relationship they exhibit (e.g., that there are four quarters in one dollar, that 50 cents are half of a dollar, that 10 dimes makes one dollar). They may be able to work (add, subtract) with those units using visual or physical representation, without understanding the place-value conception or the meaning of the decimal point. As such, their knowledge of part–whole in the money context is a separate construct from their emerging concept of part–whole in the context of the formal teaching of fractions.

#### ***At This Level, Students Can Do . . .***

Students are familiar with fraction notation, recognize common fractions in a (static) pictorial representation mostly of area shapes, and partition a whole to any given number of parts and may do it more effectively if the number of parts are even or  $2^n$  (i.e., by using repeated halving). They may have trouble if the number of parts is not even (e.g., 9, 15, or even 3) or work it through by trial and error (see examples from our cognitive interviews of the folding task). They are familiar with some decimal notation for basic fractional parts of a unit of money (say dollar), like \$0.50, \$0.25 for a quarter, \$0.10 for a dime, and \$0.05 for a nickel (i.e., benchmarks), without fully understand the place-value notation but rather see these notations as *labels* for specific amounts of money.

#### ***At This Level Students May Have Trouble With . . .***

Students have trouble partitioning a whole into an odd number of parts, accepting equal parts that are not congruent, and relating to the unit fraction as a separate unit.

### Level 2—Fraction as Unit

#### ***At This Level Students Know . . .***

Students understand the concept of a unit fraction as a separate unit that belongs and gets its meaning from the partitioned whole. They can name or use simple notation using the term *out of* as in 1 of 4 or 3 of 4 (that is, the

beginning of translating between representations—pictorial and fractional notation—occurs at this level). Evidence that this level of understanding is robust can be found in students' reaction to improper fractions; they may say that  $5/3$  cannot be, because there cannot be 5 of 3 (see the example from our cognitive interviews; see also Olive & Steffe, 2002). They can see a common fraction of  $a/b$  as built from *combining* a unit fraction  $1/b$   $a$  times. This combining of unit fractions, although it seems multiplicative (and may very well be the beginning of the multiplicative understanding), is a result of an additive structure because it is still limited to the size of the whole and is dependent on that whole. Thus, students understand at this stage addition and subtraction of fractions as joining and separating parts referring to the same whole. Their mathematical world is the whole (Hackenberg, 2007). They have emergent understanding of equivalent fractions for special cases, or benchmarks, using visual or physical models. At this level, students do not see a fraction as a single number (e.g., as a measure of . . . or a magnitude of . . .) but rather as two numbers, despite their conception of a unit fraction and their emergent understanding of equivalency.

In the decimal context, students still develop separate but parallel understanding. They understand the partition of the money unit (say a dollar) to 100 parts, and understand the meaning of any fractional part of that whole money unit (e.g., \$2.35). The basic fractional parts of the money system (quarter, dime, nickel, penny) receive a special meaning as unit decimals, which can be combined or operated upon with concrete meaning.

### ***At This Level Students Can Do . . .***

Students can work with unit fractions and can decompose a fraction into a sum of fractions with the same denominator in more than one way and record each decomposition by using an equation (e.g.,  $3/8 = 1/8 + 1/8 + 1/8$ ;  $3/8 = 1/8 + 2/8$ ;  $2\ 1/8 = 1 + 1 + 1/8 = 8/8 + 8/8 + 1/8$ ). They can justify decompositions (e.g., by using a visual fraction model). They can add and subtract fractions with like denominators and add and subtract mixed numbers where there is no need to replace the mixed number with the equivalent improper fraction (e.g.,  $2\ 1/8 + 5\ 2/8 = 7\ 3/8$ ;  $5\ 5/8 - 1\ 2/8 = 4\ 3/8$ ) by using a visual model. They can solve word problems involving addition and subtraction of fractions referring to the same whole and having like denominators (e.g., by using visual fraction models and equations to represent the problem). Similarly, students can operate with decimals in the context of money: they can decompose 75 cents to three quarters, record this as an equation ( $0.75 = 0.25 + 0.25 + 0.25$ ), and justify it using a visual model or concrete money.

Students at this level can recognize and generate simple equivalent fractions, like  $1/2 = 2/4$ , and explain why the fractions are equivalent by using a visual fraction model. They can express whole numbers as fractions and recognize fractions that are equivalent to whole numbers (e.g., express 3 in the form  $3 = 3/1$  or recognize that  $6/1 = 6$ ). They can compare two fractions with the same numerator or the same denominator by reasoning about their size and recognize that comparisons are valid only when the two fractions refer to the same whole. They can also record the results of comparisons with the symbols  $>$ ,  $=$ , or  $<$  and justify the conclusions by using a visual fraction model.

### ***At This Level Students May Have Trouble With . . .***

Although at this level students have a strong understanding of part/whole and a conception of a unit fraction, they still view fractions as two numbers. That is,  $a/b$  is viewed as  $a$  parts out of  $b$  parts, or as  $a$  times  $1/b$  fractional part of the partitioned whole. Thus, students at this level may exhibit trouble with ordering fractions (cf. Smith, 2002), dealing with improper fractions, and making a general connection between a decimal number (e.g., 0.25) and a fraction ( $1/4$ ) even though they are familiar with at least one such example (where \$0.25 is in fact a  $1/4$  of a dollar) and the language makes the connection by naming it a quarter.

When asked to build a whole out of one part (say  $1/9$ ), students at this level may be able to do it. However, they may encounter difficulty when asked to create a whole from  $6/9$  of a whole, even with manipulatives. That is, they do not yet realize that they must divide the  $6/9$  into six parts to get the unit  $1/9$  and then multiply it by 9 to form the original whole.

When asked to add or subtract fractions with unlike denominators (and without a visual model), students may revert to adding/subtracting numerator and denominators, as in the example,  $1/2 + 2/3 = 3/5$ . They may be puzzled when presented with a visual model of this problem but may find a way to show visually/qualitatively what the result should look like, even though they cannot quantify it.

### Level 3—Fraction as Single Number and Fraction as Measure

#### ***At This Level Students Know . . .***

Students can conceive of a fraction as a single number in its own right and understand the meaning of an improper fraction. The transition from viewing  $5/8$  as five parts out of eight, to viewing it as  $5 \times 1/8$ , which started to develop in the previous level, continues and crystallizes here with the understanding that one can iterate this unit whatever number of times, even beyond the original partitioned whole. Thus, at this level students can hold three levels of units (cf. Hackenberg, 2007; Steffe, 2002) in their head: the whole, the unit fraction, and the improper fraction/mixed number. At this level, students have a notion of *magnitude* attached to the fraction; they understand the concept of equivalent fractions and that different labels and notations can refer to the same magnitude/measure/value. A fraction as a measure is established.

In the decimal arena, students are ready to conceptualize decimals as numbers detached from money. They understand that the decimal notation expands the whole number notation, and they see it in the measurement context where a measure of an item is denoted as 2.45 cm, and so forth.

#### ***At This Level Students Can Do . . .***

Students are able to order fractions, order decimals, convert between basic fractions and basic decimals (i.e., benchmark fractions), find equivalent fractions, convert from improper fractions to mixed numbers, find a common denominator for two different fractions, add and subtract fractions with like denominator without a visual model and also add and subtract fractions with unlike denominators using a common denominator, and multiply a whole number with a fraction (generalizing the iterating process from the previous level that was done with unit fraction to be done also with any fraction).

#### ***At This Level Students May Have Trouble With . . .***

Students still show difficulty with ordering all fractions and decimals on the same number line and mostly have trouble with fractions like  $1/3$ ,  $1/6$ , etc. They do not yet have fluency with the new procedures, so they may have trouble working when numbers are less familiar. They have not yet acquired the concept of density and the concept that between each two fractions can be found another one.

### Level 4—Representational Fluency

#### ***At This Level Students Know . . .***

Students have a concept of multiple notations/representations of a number. That is, they have mastered the notion that a referent (value) can be expressed in different ways (fraction, decimal, simplified fraction, mixed number). Students at this level have a strong conception of addition and subtraction of fractions and decimals, their meaning, and their application in different context. They can model and contextualize addition and subtraction in a word problem. They begin to have an early multiplicative structure, and they can identify multiplicative relations and patterns to use in problem solving. They understand partitioning in a deeper way that allows them to use it even in complex number combinations and exhaustively (partition the remainder). They understand the concept of a fraction as operator and thus see a mapping between a fraction and a decimal (or percentage), for example,  $(1/4) \times 5$  read as a quarter of 5 or 25% of 5. Students have an increased symbolic fluency over that of Level 3.

#### ***At This Level Students Can Do . . .***

Students can partition exhaustively ( $10/3 = 3.33333 \dots$ ) and not be satisfied with a remainder, and they can compute a fraction of an amount, as in  $(1/4) \times 5$ , or 0.25 (25%) of 5. They can flexibly move between equivalent expressions, like  $3 \times (2/5)$  equivalent to  $6 \times (1/5)$ , and solve word problems involving multiplication of a fraction by a whole number by using visual fraction models and equations to represent the problem.

***At This Level Students May Have Trouble With . . .***

Students have trouble contextualizing and modeling division problems with fractions and decimals. They have trouble generalizing from concrete problem solving strategies to general strategies; for example, although they can flexibly move between equivalent expressions, like  $3 \times (2/5)$  equivalent to  $6 \times (1/5)$ , they would not see the general case where  $n \times (a/b) = (n \times a)/b$  OR  $k \times (a/b) = k \times a (1/b)$ .

**Level 5—General Model of Rational Number*****At This Level Students Know . . .***

Students understand the multiple faces of fractions, and specialize and generalize across contexts (conceptual depth and breadth/contextual fluency). That is, some contexts require conceiving the fraction as a single number, whereas others may require viewing it as a relation between two quantities (i.e., in ratio situations). Students at this level have a strong connection between the numerical representation (the fraction, the decimal, an expression that includes both) and the context. They can decontextualize from complex word problems to expressions and equations as well as find a context appropriate for an equation or expression involving a rational number of any sort, including division of fractions.

***At This Level Students Can Do . . .***

Students are able to make general statements about the truth and falseness of rules or algorithms and can explain their meaning; they can use shortcuts in an equipartitioning situation where  $a$  items are divided to  $b$  people and know without computing that the part each one gets is equal to  $a/b$  (the quotient). They can find a context to describe a division of a fraction by a fraction, or a mixed number by a fraction.

***At This Level Students May Have Trouble With . . .***

This is a mastery level, hence no trouble.

Table 3 summarizes the progression in terms of conceptual understandings and their relationships to fraction and decimal notation.

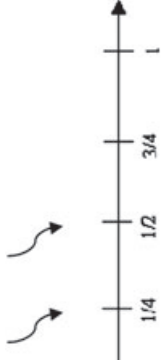
**Evidence From Cognitive Interviews**

Fourteen students in Grades 3–5 (five students, four students, and five students, respectively; nine female and five male) were led through semistructured cognitive interviews with sample tasks designed to elicit evidence of understanding on the basis of the drafted LP. We also interviewed two elementary mathematics teachers with the same tasks, asking them to both solve the problems and to simulate a hypothetical student responding to the same tasks. The students' cognitive interviews were used to test our hypothetical LP, and slight changes were made to both the progression and the tasks as we continued to conduct the interviews. Each participant (student or teacher) was invited to one individual 40–60-minute session, and the session was videotaped. We used eight tasks; each pair of tasks was designed to target a transition between two adjacent levels in the LP (4 transitions). The tasks were presented in the same order for all students and teachers (organized by the levels and our assumed difficulty estimates). Although the tasks were ordered by hypothesized difficulty, if a student struggled on a task, we still had him or her progress to the next task, as we found that some of the tasks were harder than we had anticipated. For two students in Grade 3, after struggling with several tasks in a row, we did not include the last two tasks (that addressed Levels 4 and 5).

Below, we outline five sample tasks and some evidence garnered from the cognitive interviews. We found through the interviews that some of the tasks were more informative than others, and thus, we provide an analysis only of those tasks that ultimately informed the LPs and/or further task development. Due to the small sample size and exploratory nature of the interviews, no statistical results are presented, just general qualitative observations based on the set of interviews.



Table 3 Rational Number Learning Progression Summary

Level	Conceptual understanding	Fraction example	Decimal example
Prior knowledge	The concept of half; halving or splitting into two equal parts	Breaking a cookie to share between two people	
Half and halving Level 1	The beginning of part/whole and part/group understanding; repeated halving, and equipartitioning into number of parts 2 <sup>n</sup>	$1/2$ , $1/4$ , $1/8$	0.5 as equal to half; 0.25 is a quarter In context (e.g., money)
Early part/whole understanding		In context	
Level 2	The establishment of part/whole concept of a fraction; equipartitioning with all numbers	Unit fractions: $1/2$ ; $1/3$ ; $1/4$ ; $1/5$ ; $1/6$ ; $1/7$ ; etc.	\$2.50 means 2 and a half dollars Basic unit decimals
Fraction as unit	Unit fraction concept and common fractions smaller than one whole (proper functions)	Common fractions: $2/5$ conceived as 2 times $1/5$	$0.01 = \text{penny}$ ; $0.05 = \text{nickel}$ ; $0.10 = \text{dime}$ ; $0.25 = \text{quarter}$ Nonbenchmark decimals, such as \$2.35
Level 3	The shift to the concept of fraction as a single number; a number-line representation	0.25 0.5	
A fraction as a single number	Early integration of fraction and decimal notation (benchmarks); fraction as a measure		
Level 4	Smooth translating between different notations of rational numbers	Transform $2/3$ into $2/5$ multiplicatively	Convert $5/8$ into decimal form, in any algorithm
Representational fluency	Established multiplication of fractions with fractions, i.e., partitioning of a partitioning (partitioning a third into fourths results in $1/12$ size parts); fraction as an operator Convert smoothly between fractions and decimals General model of fraction; multiplicative structure; contextual fluency; fraction as quotient		
Level 5		What is a situation in which you would divide $3/4$ by $5/8$ ?	If three pounds of pasta salad cost \$4.50, how much pasta salad can you buy for \$6.00?
General model of rational number	Explain fraction/decimal multiplication in context Explain fraction division in context (quotative division concept)		

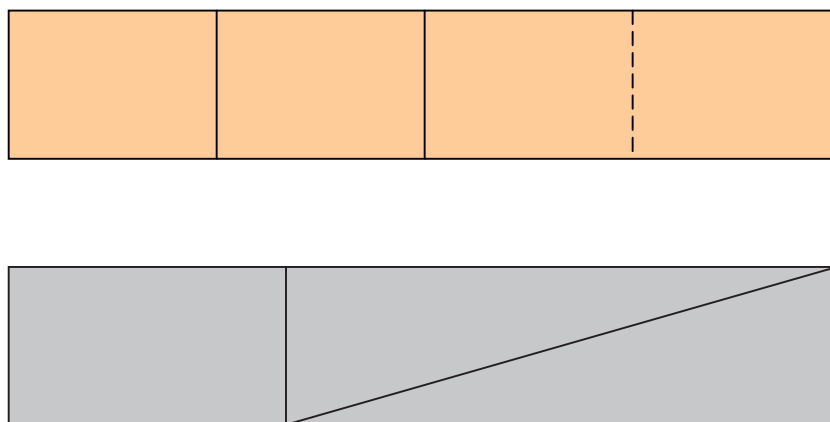


Figure 3 Partitionings presented to students: the folding task.

### Early Equipartitioning—The Folding Task (Targeting the Distinction Between Levels 1 and 2)

In the folding task, students are asked to fold strips of paper, one at a time. First the student is asked to fold a strip into fourths. The second strip of paper is then given, and the student is asked to fold it into thirds. After that, the student is presented with two additional partitionings (presumably from another student) and is asked whether they are valid thirds or not. The first partitioning is of the strip to fourths and then cutting one part to be left with three equal parts (see the top of Figure 3), and the second is a valid partition to thirds where the parts are not congruent shapes (see the bottom of Figure 3).

### Research Questions

The folding task was designed to address Levels 1 and 2 of the LP, which speak to students' understandings of part and whole relationships, most easily with partitioning into an even number of parts. At these levels, they are beginning to understand the notion of equipartitioning, but at Level 1, they are not necessarily applying it to area of the parts but rather to visual/shape similarity between parts. The notion of the whole may also be shaky in this stage, where students at Level 1 may allow changing the whole and seeing the part as part of a new whole. Our four research questions related to the four requests in the task are as follows:

1. What are the different strategies students use when folding into fourths? Do students use repeated halving (as mentioned in the literature)? Are these students the ones who also do better in other tasks compared to those who fold into four pieces serially?
2. Is folding into thirds more difficult than folding into fourths? What are the strategies students are using for that operation (repeated halving is not possible)?
3. How do students perceive the invalid way of cutting (and thus changing the whole)?
4. How do students perceive the valid way of creating incongruent yet equal parts?

### Evidence From Cognitive Interviews

#### *Strategies for Creating Equipartitions*

The first research question concerned what types of strategies students would use when creating fourths out of the strip of paper. This type of question has been asked in previous literature (Empson & Turner, 2006). However, we wished to use it to help pinpoint student placement on the LP across tasks.

Ultimately, two different strategies emerged for creating fourths. Most students (10 of 14) showed repeated halving; eight folded the second fold in the same direction as the first (two folds vertically), one made the second fold at a 90-degree angle from the first (one fold vertically and the second horizontally), and another showed both ways. The remaining four students showed *serial folding*, by which they first had to estimate the needed size, make one fold, and serially make the other folds based on the first, while checking and adjusting if needed. Of these students, two made four folds that resulted in five parts and were puzzled with the outcome when they opened the folded strip. All four students needed several trials

to achieve success. For all four, the next request to fold into thirds seemed to be easier, while for the 10 who used repeated halving, thirds were more difficult.

The request to fold into thirds yielded the same strategy, serial folding, from almost all students (12 of 14). Five students efficiently estimated the needed size and succeeded in the first or second trial; the other seven students needed more trials (one of whom adjusted in the opposite direction than needed; that is, when the first trial resulted in partitioning less than the whole strip, her adjustment was to make smaller folds instead of larger ones). Two students demonstrated unique approaches: one third-grade boy actually used the strategy of folding into four parts and then cut one part and presented us with the three parts of the new cut strip (which rendered our next task unnecessary). Finally, one fourth-grade student used approximation by repeated halving: the first fold into fourths, then the two middle ones in half, then again in half, and finally the student pointed to the two lines as the  $1/3$  and  $2/3$  (actually they were  $5/16$  and  $11/16$ ). This last strategy is in fact a generalization of the repeated halving and took quite a long time and effort from the student, who was seeking a systematic approach and did not give up.

### *Recognizing Equal Parts*

When presented with the invalid cutting off of the fourth part of the fourths to be left with three parts, the reaction from most students was laughter, though many also were not sure if this cutting was allowed or not. Six of the 13 students said that these parts are thirds, because there are three equal parts. One student was excluded since this was his own strategy for creating thirds. After additional probing (“Did the student provide thirds of *the initial strip*?”), two of those students said that these parts are actually fourths of the original strip, although thirds of the new cut strip. Three students argued that this was cheating and the new parts are quarters. One of those students said that the new strip is three quarters. Four students said that the new parts are thirds and quarters at the same time, but of different wholes. Interestingly, one of the teachers interviewed (but not the other) did not accept this explanation as valid even after several probes. She claimed that you cannot see the same part one time as a third and one time as a fourth; rather, it is a fourth due to the way it was created.

The incongruent-but-equal partitioning had interesting results as not only nine of 14 students, but also one teacher claimed that the unconventional partitioning of the strip to three parts does not yield thirds. They relied on the proposition that the parts are not equal and therefore they are not thirds. Further probing to justify why they were not equal, elicited in two students’ ways to check whether the areas of the parts were equal or not. In other cases, the answer remained that “their shape is not equal so they cannot be equal.” A third-grade girl said, “They wouldn’t be happy; it is not fair sharing.” However, some students did say that the parts are equal, with one saying it seemed equal but he could not really tell, one saying she did not know how to show they are equal but probably “by moving things around,” and two by giving a demonstration of equal area by folding into thirds and then on the diagonal line.

These results support the hypothesized first two levels of the LP, and the findings we observed and report here are similar to those that are documented in the literature. The cognitive interviews with this task yield variations in the strategies students could use, hence proving this task to be informative for later development of diagnostic tasks targeted at the lower end of the LP.

## **Number-Line Conception: The Living on One Street Task (Targeting the Distinction Between Levels 2 and 3)**

### ***Research Questions***

The living on one street task, shown in Figure 4, targets Levels 2 and 3 of the LP and the transition from separate concepts of fractions and decimals to the beginning of making the connection between them via benchmarks and on the number-line model (e.g., 0.5, 0.75). Also measured in this task: the understanding of improper fractions, number line beyond  $[0,1]$ , understanding of order, equivalence and sensitivity to equal intervals, as well as perceiving a fraction as a single number. Our research questions are as follows:

1. Without directly defining the street as a number line, does this task elicit a mental number-line representation? What evidence of mental line representations are shown in response to this task?
2. How do students make sense of mixed notation?
3. How would students contend with the improper fractions and number line that is larger than  $[0,1]$ ?

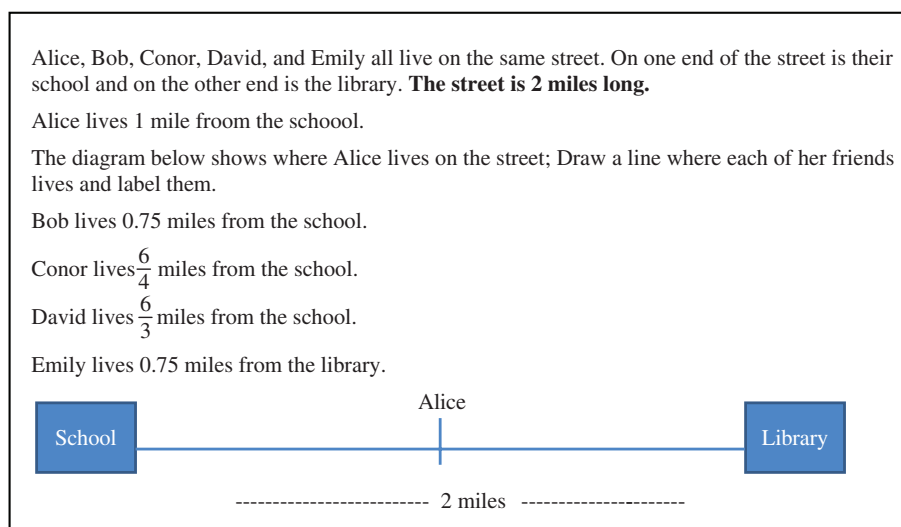


Figure 4 Living on one street task.

### Evidence From Cognitive Interviews

#### Mental Number Line

Four of 14 students divided the 2-mile street into quarters of a mile, marking them as 0.25, 0.5, 0.75, [Alice's house], 1.25, 1.5, 1.75, and used a strategy of translating all the distances presented in fractions into decimals (i.e.,  $6/4 = 1.5$ ;  $6/3 = 2$ ). This demonstrated an explicit understanding of the street as a number-line representation. For the other 10 students, this behavior was not observed. This conclusion is not to say that they did not perceive of the street as a number line, but that there was no explicit action or verbalization to demonstrate such a representation.

#### Notations

As mentioned above, four students out of the 14 showed no problem with the mixed notation. They converted the fractions into decimals and then worked from there to order the numbers. However, the rest of the students showed some difficulties.

Three students showed explicit difficulty in reading an improper fraction. One student read  $6/4$  as six-four and placed it after  $6/3$  which he read as 63, saying that these values are “one point apart,” hence, placing six-three at approximately a 0.7 distance from school, and six-four at approximately 0.8. The second student read 4 out of 6, saying “there cannot be 6 out of 4, so it must be 4 out of 6.” The third student read them as  $3/6$  (three sixths) and  $4/6$  (four sixths), divided the distance into sixths and placed them in the respective locations where  $3/6$  and  $4/6$  would belong. These three students seem to have a  $[0,1]$  number line, as they treated each mile separately: all three students successfully placed Bob and Emily (0.75 from the school and the library respectively), using *mirroring* techniques (one student measured with her fingers from each end point—school or library—to make sure she was placing them at the same distance). Since they also treated the improper fractions as fractions smaller than 1 (i.e.,  $4/6$  instead of  $6/4$ , and  $3/6$  instead of  $6/3$ ) and partitioned the first mile into sixths, this behavior seemed enough evidence to infer a representation limited to  $[0,1]$ .

The seven remaining students seemed to struggle in other ways, and although it seemed that they knew how to read aloud both improper fractions and decimals, they showed difficulty in interpreting these numbers. For example, they placed and moved the houses several times, seeming not exactly sure where they belonged. In some cases, it was evident that they could recognize that  $6/4$  is actually  $1\frac{1}{2}$  miles, but the proximity to Emily's house at 0.75 mile from the other direction was confusing for them. They weren't sure whether to place the 1.5 before or after 0.75 from the library, which is in fact 1.25 from the school. This task, in fact, required them to be aware of two number lines, or to subtract the 0.75 from the 2 miles, in order to be able to compare the two numbers on the same number line. One particular student specifically said that she did not know what to do and refused to place the house.

Overall, the number of students who placed each house correctly was

- Bob [0.75 from school] — 10 students,
- Conor [6/4 from school] — seven students,
- David [6/3 from school] — four students, and
- Emily [0.75 from library] — nine students.

### ***Dealing With a Number Line Larger Than [0,1]***

The observations from this task provided surprising evidence of how students who are not familiar with improper fractions (thus not yet Level 3) find creative solutions in this context. The fact that the task did not directly define the street as a number line allowed variation in strategy use, with some students perceiving the whole street as a  $[0,1]$  number line. This whole-street perception again supports the assumption that at early stages of understanding, students may have a representation of a number line limited to the range of  $[0,1]$ .

### ***Final Remarks***

Most of the students when asked at the end of the interview which task they liked the most mentioned this task, probably due to the interactive nature. We further developed a task model for formative assessment based on this preliminary task.

### **Fraction as Ratio Task: The Dividing Bananas Task (Targeting the Distinction Between Levels 2–5)**

The dividing bananas task was originally inspired by a Fosnot and Dolk (2002) activity in which students were asked to split sandwiches on a class trip. In the dividing bananas task, students are told the story as shown in Figure 5.

### ***Research Questions***

The following research questions were developed based on the dividing bananas task:

1. What strategies do students use when dividing into fair shares? How would they compare parts without actually computing them?
2. Do students recognize that they can figure out the amount of bananas per student by making the numerator the number of bananas and the denominator the number of students?

### ***Evidence From Cognitive Interviews***

The dividing bananas task turned out to be much more difficult than anticipated, as none of the 14 students recognized that they could figure out the amount of banana per student by making the numerator the number of bananas and the denominator the number of students (or by dividing number of bananas by number of students).

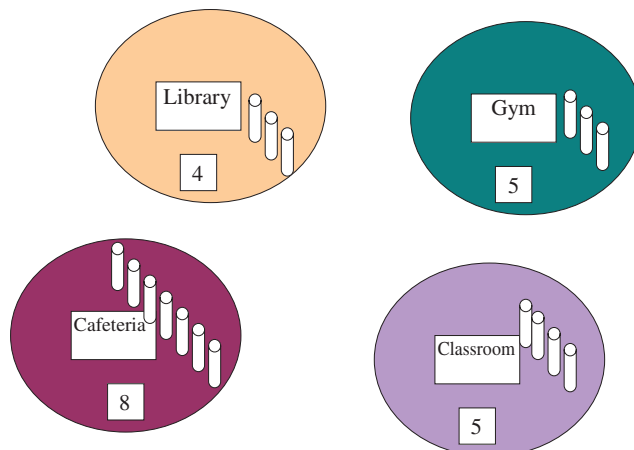
Nine of the 14 students recognized qualitatively that three bananas and five students results in less bananas per student than three bananas and four students because the same amount of food is divided by more people, but they were unable to quantify the ratios or the differences. Additionally, six of the 14 students believed that the three rooms in which there was one more student per banana would all have the same amount of bananas per student because of the differences in the two numbers, demonstrating additive but not multiplicative thinking. None of the students recognized that they could figure out the amount of bananas per student by making the numerator the number of bananas and the denominator the number of students.

Due to the difficulty of the task, it underwent a number of modifications in which we broke up the task into mini sub-tasks with much more scaffolding. Hence, we were unable to provide more quantitative results from this task since students saw different versions of the task and were asked different questions. We did, however, observe a variety of responses, and in several interviews, much time was spent on this task, oftentimes without reaching an answer. Additionally, from the three main findings above, as well as other evidence elicited from the questioning of individual students, we found this task to provide rich information in eliciting several strategies that are relevant to different levels in the progression. Thus, we further continued to develop a task model based on the idea of fair sharing that is explored here (see Appendix A).



On a rainy day at school, students could choose between four different activities: reading at the library, soccer at the gym, dancing in the cafeteria, or singing in the classroom. Students could choose which room to go to. In each room, there were bananas to be given to students for a snack.

The organizers guessed how many students would choose each room, and divided the bananas this way [place bananas]. For example: they thought that the cafeteria would be the most popular and therefore left seven bananas in that room.



1. These numbers show how many students showed up to each room [Place numerals]. More kids showed up to school than were anticipated. Some kids complained that not all of them got a fair share of bananas. We are going to figure out if they were right. Let's start by comparing the library and the gym. Did these students get the same amount of bananas?
  2. Now let's add the classroom. Did these students get more or less bananas than the kids in the library?
  3. Another student said that the library, the cafeteria, and the classroom all had the same amount because there was one more student than the number of bananas. Do you agree?
- At the end of the task, ask: "What would be a fair share of bananas for all of the students?"

Figure 5 The dividing bananas task

### Misconception of Adding Numerators and Adding Denominators—"Is It True?" Task (Targeting the Distinction Between Levels 2 and 3)

#### Research Questions

The following research questions were developed based on the "is it true?" task (shown in Figure 6).

1. How do students react to this question, and what strategies do they use to support their reasoning?
2. Does the performance on this task support the evidence observed in previous tasks targeted at these two levels (Levels 2 and 3; the number-line task)?

#### Evidence From Cognitive Interviews

Eight of 14 students correctly stated that this equation was not true, with seven of them providing evidence on the basis of a common denominator. The eighth student explained his reasoning by using *magnitude* consideration. He said: "2/3 is more than 1/2, so that means that 1/2 plus the 2/3 has to be more than 1."

Of the remaining six, one said that she did not know and provided no further explanation, while the other five said that this equation was true, and showed that  $1 + 2 = 3$  and  $2 + 3 = 5$ , revealing an additive misconception.

The evidence from this task supports earlier findings in the literature regarding a common misconception among students who are not yet at the stage of perceiving a fraction as a single number. We also observed that students who

Another student wrote this:

$$\frac{1}{2} + \frac{2}{3} = \frac{3}{5}$$

Is this true?

How do you know? Show me any way that you can.

Probe the student for various ways to show why this is true or not. [Drawing, number line, example in context, general arguments, etc.]

Figure 6 Is this true? task.

**Materials:** Flashcards with numbers

**Step 1:** Only decimals cards. Ask student to order the cards from the smallest to the largest and to put aside the numbers that he or she has never seen before.

0.75   2.5   1.41   0.33333...   0.5   0.25   1.09

**Step 2:** Present the fractions cards. Ask student to place the other numbers in between the ordered decimals.

$\frac{3}{4}$     $\frac{2}{3}$     $\frac{1}{2}$     $\frac{5}{2}$     $\frac{5}{5}$    1

$\frac{4}{3}$     $\frac{3}{10}$     $1\frac{1}{10}$     $\frac{2}{5}$     $\frac{1}{3}$

**Step 3** if  $\frac{3}{4}$  and 1 are next to each other: Ask, “Can you think of a number in between these two numbers?” and point to  $\frac{3}{4}$  and 1.

Figure 7 Ordering cards task.

revealed this misconception showed difficulties in the number-line task. This association provided good support of the LP in its distinguishing of Level 2 from Level 3.

### Advanced Number-Line Conception—Ordering Cards Task (Targeting the Distinction Between Levels 3 and 4)

#### Research Questions

The following research questions were developed based on the ordering cards task (as shown in Figure 7).

1. How do students order decimals with different numbers of digits after the decimal point?
2. How do students order fractions with different denominators?
3. Can students order both fractions and decimals on the same number line or same chain of numbers? What strategies do they use? Which difficulties will they encounter?

#### Evidence From Cognitive Interviews

The following evidence was gathered from cognitive interviews for ordering decimals and fractions and decimals together.

##### Ordering Decimals

Though ordering of decimals was anticipated to be a difficult task due to curricular emphasis, students did surprisingly well. Six of 14 students ordered all of the decimal cards correctly. One of these students used a strategy of adding zeros to the decimals, such that 0.25 and 0.50 have the same number of digits after the decimal point. The remaining eight students made a partially correct order, with the following difficulties:

1. Placing 0.5 before 0.25 (seven students)
2. Not familiar with 0.3333... (five students)
3. Placing 1.41 before 1.09 saying that 1.09 is almost 2 (one student)

### *Fractions and Decimals Together*

Four students correctly added the fractions to the already ordered decimals, and all four used a strategy of converting fractions to decimals.

Most other students found this task quite challenging. Five students did not know how to address this task at all so we revised the request and asked them to order the fractions separately (we report on those next). The rest struggled and kept moving cards around. Strategies that were identified included comparing to half (e.g., asking for each new card: “Is this more or less than half?”), but most students relied on paired comparisons of the presented values.

Some of the difficulties were as follows:

- Not knowing where to put  $2/3$  and  $1/3$  in the overall order
- Not knowing how to order  $1/3$  and  $3/10$
- Not knowing how to order  $4/3$  and  $1.41$
- Not differentiating  $2/5$  and  $5/2$  (e.g., one student placed  $1/2$  under  $0.5$  and then put  $5/5$  under  $0.25$  and then did not know what to do with  $5/2$  and so placed it in between; another student just put them one under the other, as if they were equivalent)
- Placing only a few benchmarks ( $1/2$ ;  $1$ ;  $1\ 1/10$ )

As noted above, adding the fractions into the already ordered decimals was a task that five students did not know how to address at all, and hence, we revised the task on the spot and asked them to order the fractions separately. Some interesting ordering of fractions emerged:

1.  $1/2$ ,  $1/3$ ,  $2/3$ ,  $2/5$ ,  $3/4$ ,  $3/10$ ,  $5/2$ ,  $5/5$ ,  $1$ ,  $1\ 1/10$  (two students)
2.  $1/2$ ,  $1/3$ ,  $1$ ,  $2/3$ ,  $2/5$ ,  $3/4$ ,  $3/10$ ,  $5/2$ ,  $5/5$  (two students)
3.  $1/2$ ,  $5/2$ ,  $1/3$ ,  $2/3$ ,  $4/3$ ,  $3/4$ ,  $2/5$ ,  $3/10$  (one student)

Note that the first two orders are a result of the same strategy, whereby the student ordered the fractions by the numerator (as if reading them as decimals  $1.2$ ,  $1.3$ ,  $2.3$ ,  $2.5$ , . . .) with some confusing where to place  $1$ . The first two students placed it at the end, while two other students placed it after the unit fractions. The fifth student ordered the fractions by the denominator, or maybe by families of fractions, still holding to the part–whole conception.

When asked after finishing that task to combine the two orders (the decimals and the fractions), these five students all said that either they did not know how or it was not possible to do. One student gave an interesting argument: “Those are decimals and money, and those are ‘out of’ parts—you cannot put them together, they have different meaning.” These findings reiterate the issues in the literature review that students are taught decimals and fractions separately until at least the fifth grade.

### **Conclusions**

The students’ cognitive interviews largely confirmed our LP and the tasks described above allowed us to make judgments of student placement in the LP. Additionally, the interviews showed some students to perform in unexpected ways, suggesting modifications that may need to be made to the progression. Two of the tasks were later elaborated into task models and eventually CBAL Mathematics tasks. We are also able to use these student responses in the creation of teacher materials for use with the associated tasks.

### **Summary**

Prior work on the CBAL math competency model resulted in an initial competency model for middle school grades with several LPs that elaborate central ideas in the competency model and that provide a basis for connecting summative and formative assessment (cf. Graf, 2009). This work resulted in creating a competency model for Grades 3–5 that is based on both the middle school competency model and the CCSS. We also developed a LP for rational numbers based on an extensive literature review, consultations with members of the CBAL math team and other related research staff at ETS, expertise of an external advisory panel in math education and cognitive psychology, and the use of small-scale cognitive interviews with students and teachers. This work expands the CBAL library of LPs.

Elementary mathematical understanding, specifically that of rational numbers, is viewed as fundamental and critical to developing future knowledge and skill in middle and high school mathematics and therefore essential for success in

the 21st century world. This work provides the theoretical background in order to produce assessment tasks for Grades 3–5, via the task models that grew out of the conceptual and exploratory work. The rational number LP presented here comprises an innovative combining of two strands that are often taught separately in the elementary grades: fractions and decimals. The CCSS document also includes each strand as part of a different topic. Yet, we showed here the strong conceptual similarities and connection that both strands share, and several of the tasks we studied in the cognitive interviews made explicit this conceptual connection.

More research needs to be done to empirically validate the progression herein proposed. Yet, the ideas behind creating a concept-based progression may stimulate new directions in task development.

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### Note

- 1 The Advisory Panel conference was held on September 13–14 at Educational Testing Service. The members of the panel were (a) Susan Empson of the University of Texas at Austin, expert in fractions, multiplicative structures, and the CCSS; (b) Amy Hackenberg of Indiana University, expert in student fractional schema; (c) Carolyn Maher of Rutgers University, expert in the development of mathematical thinking and reasoning; and (d) Yukari Okamoto of University of California at Santa Barbara, expert in fractions and mathematical representations.

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## Appendix A

### The Development of Task Models

Derived from the initial tasks we used for the cognitive interviews, we further developed task models as a basis for CBAL scenario-based assessment. To make the connection to assessment design more concrete, we defined types of evidence that can be used to make inferences about those levels and types of prompts that can be embedded within CBAL task sets to gather this evidence.

The following are our considerations in developing the task models:

1. Links to the CCSS
2. Links to the rational number LP
3. Links to the elementary school competency model
4. Defining the mathematical characteristics, including variables that have to be constant across parallel tasks and variables that can change

Thus far we developed four task models that cover the spectrum of the levels of the progression: a task model for sharing and equipartitioning, a task model for number-line representation, a task model for addition and subtraction with rational numbers, and a task model for multiplication and division with rational numbers. We outline here one of the task models that derived directly from a task that was studied in the cognitive interviews.

### A Task Model Involving Rational Number Reasoning in a Sharing Context

This task model targets standards for Grades 3 and 4 but is likely suitable for Grade 5 as well, perhaps in the fall of the school year.

#### General Description

This description specifies how to develop tasks that allow students to explore part/whole relationships and the magnitude of fractions with differing numerators and denominators.

The design of this task model is motivated by the CBAL elementary mathematics competency model, the CBAL rational number LP, and the CCSS in Grades 3–5.

The following are excerpts from the CCSS in mathematics for fractions in Grades 3–5 (NGA & CCSSO, 2010):

### ***Develop Understanding of Fractions as Numbers***

- 3.NF.1. Understand a fraction  $1/b$  as the quantity formed by 1 part when  $a$  whole is partitioned into  $b$  equal parts; understand a fraction  $a/b$  as the quantity formed by  $a$  parts of size  $1/b$ .
- 3.NF.3. Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.

### ***Extend Understanding of Fraction Equivalence and Ordering***

- 4.NF.1. Explain why a fraction  $a/b$  is equivalent to a fraction  $(n \times a)/(n \times b)$  by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.
- 4.NF.2. Compare two fractions with different numerators and different denominators (e.g., by creating common denominators or numerators) or by comparing to a benchmark fraction such as  $1/2$ . Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols  $>$ ,  $=$ , or  $<$ , and justify the conclusions (e.g., by using a visual fraction model).

It should be noted that this task model may serve as a model for a series of shorter tasks, but is presented here as a single task for clarity of the progression of the situation.

### **Mathematical Characterization**

This task model specifies that there are multiple instances of an entity (each differing in magnitude), each of which is to be divided into several parts. Students will need to compare the relative sizes of the parts as well as determine what the overall size of the parts would be if all of the entities were combined and split into the number of parts needed.

This task model targets Levels 1–3 of the rational number LP, with an emphasis on Levels 1 and 2:

1. Part/whole concept with denominators  $2^x$
2. Part/whole concept with all denominators
3. Single number concept

### **Targeted Standard 1**

The first targeted standard is 3.NF.1 (in Grade 3, the fraction  $1/b$  is limited to denominators 2, 3, 4, 6, 8). The initial fraction represented is that of  $1/4$ , explicitly, one object divided by four people (as opposed to two objects divided by eight people). For formative assessment, we may wish to start with one object divided by two people as most students in the third grade should recognize this as  $1/2$  and it would lay the groundwork for a good understanding of the situation. However, this may be considered too much scaffolding for summative assessment. This initial item ties directly to the first half of the CCSS standard (part/whole understanding of unit fractions). We next repeat this situation for one object divided by six people to target the same standard with a denominator that is not represented by  $1/2^x$ , as this has been shown to be more difficult. This targets Level 1 of the rational number LP.

Next, we target the second half of the standard (part/whole understanding of fractions with numerator  $>1$ ) by introducing more than one object to be shared. Explicitly, we begin with a number of objects and people that represent the fraction  $1/2$  as a springboard, for instance, eight people and four objects. We may wish here to add more items of this type that represent  $\frac{1}{2^x}$ ,  $\frac{1}{2x}$ , and  $\frac{1}{2x+1}$  (that last only if targeted above Grade 3) before moving onto  $\frac{a}{2^x}$ ,  $\frac{a}{2x}$ , and  $\frac{a}{2x+1}$  (again, the last only if targeted above Grade 3). This targets the Level 1-to-2 transition of the rational number LP.

### **Targeted Standard 2 and Targeted Standard 3**

The second targeted standard is 3.NF.3, specifically these concepts (NGA & CCSSO, 2010):



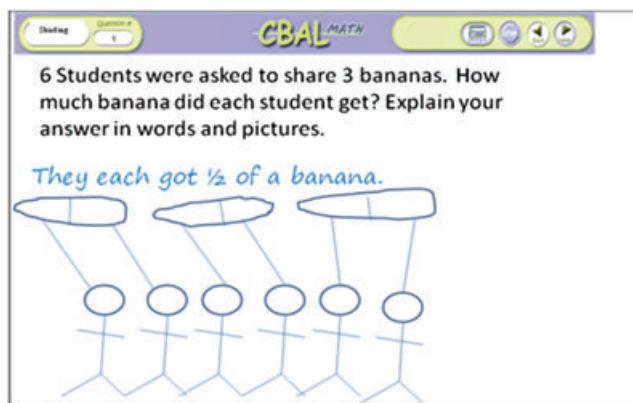


Figure A1 Example of a student answer.

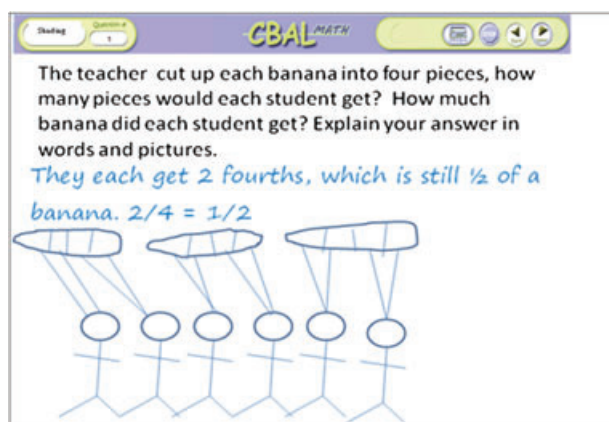


Figure A2 Example of a student answer.

- Understand two fractions as equivalent (equal) if they are the same size or the same point on a number line.
- Recognize and generate simple equivalent fractions (e.g.,  $1/2 = 2/4$ ,  $4/6 = 2/3$ ). Explain why the fractions are equivalent (e.g., by using a visual fraction model).

The third targeted standard is 4.NF.1. Now that students have named individual fractions that are represented by various scenarios, we move onto comparisons that target both 3.NF.3 and 4.NF.1. First, students are asked to compare two equivalent situations:  $\frac{a}{b}$  and  $\frac{ax}{bx}$ . They will use visual models to explain these comparisons (hand-drawn, if technologically possible, or by arranging figures). For example, see Figures A1 through A3. This targets the Level 1-to-2 transition of the rational number LP.

## Targeted Standard 2 (Part 2)

For the second targeted standard, 3.NF.3., specifically we targeted the following (NGA & CCSSO, 2010):

- Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols  $>$ ,  $=$ , or  $<$ , and justify the conclusion (e.g., by using a visual fraction model).

The situation is now changed only in the number of objects or the number of people, but not both. Students first reason intuitively about the situation (e.g., if I have the same number of objects, but more people to share it, each person must get less) and then consider the mathematics of the fractions involved to demonstrate that  $\frac{a}{d} < \frac{b}{d}$  if  $a < b$ . Similarly, if I have more objects and the same number of people to share it, everyone will get more, so,  $\frac{a}{d} > \frac{a}{e}$  if  $d < e$ . This targets the Level 1-to-2 transition of the rational number LP.



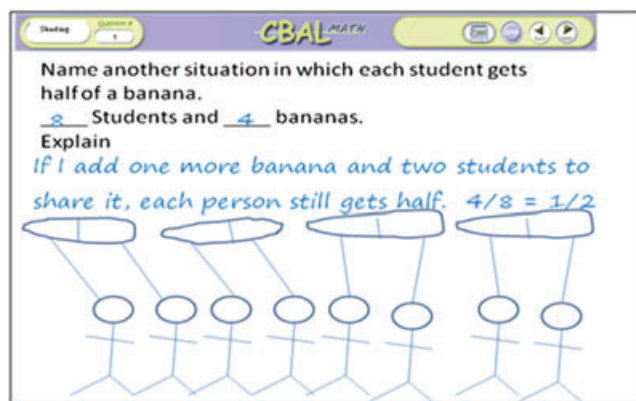


Figure A3 Example of a student answer.

### Targeted Standard 4

The fourth targeted standard is 4.NF.2. For this standard, we reach a situation in which the number of objects and the number of people are both changed. Students must compare fractions with unlike numerators and denominators. This begins with comparison of various fractions  $\frac{a}{a+1}$  as a common misconception is that these fractions are equal. Thus, comparing them outright at the start addresses both this misconception as well as general methods of comparison. This targets the Level 2 to Level 3 transition of the rational number LP.

Students may be able to solve the items by converting the fractional representations into decimal representations, though the ordering of the fractions would be obvious to those with a more general model (for instance,  $\frac{7}{8}$  vs.  $\frac{5}{6}$ ), such that information on which students do choose to convert to decimals can suggest the level of the LP that a student has reached.

### Notes When Generating Examples

While creating examples, keep in mind that the entity should be something that does not lose value when split. For instance, apples are typically eaten in slices, so two halves of an apple is equivalent to one whole apple. If a cookie were chosen, for instance, the context could be distracting from the mathematics as students may argue that sevenths of a cookie are too small eat and therefore just crumbs. Variations could include both discrete (e.g., apples) and continuous (e.g., milk) quantities. It is advised to avoid certain units, such as yards or feet in which students could convert to a smaller unit (inches) to avoid the use of fractions at all.

### Variations

There are a number of variations on this task model that could target other standards of the CCSS, as well as higher levels of the LP.

For example, one variation is utilizing numbers that require or favor division/decimal conversion in order to solve (for instance  $\frac{5}{332}$  vs.  $\frac{7}{379}$ ). This variation would also target Level 3 of the Learning Progression, but in the decimal notational field. Another variation is to begin with the fractional amount of an entity and, given the number of students or number of total entities, solve for the other quantity.

It is also possible to extend the situation to cover addition and subtraction of fractions (4.NF.3, 5.NF.1, and 5.NF.2) and multiplication of a fraction by a whole number (4.NF.4)

## Appendix B

### Relating the Levels in the Learning Progression to Common Core State Standards

Table B1 outlines specific standards in the CCSS and their relationship to the LP. Note that there is not always a direct correspondence as some standards may have nuances, which can fall in two (or more) different levels of the progression.

Table B1: Learning Progression (LLP) Level and the Relevant Common Core State Standard (CCSS)

Level in the LP	Relevant CCSS (NGA & CCSSO, 2010)
<p>Level 1: The beginning of part/whole and part/group understanding; <b>repeated halving</b> and equipartitioning into number of parts <math>2^n</math>; separate understandings of a part/whole relation as exhibited in fractions in equipartitioning context and in decimals as related to money</p> <p>Level 2: The establishment of part/whole concept of a fraction; equipartitioning with all numbers; <b>unit fraction concept</b> and common fractions smaller than one whole (proper fractions); separate understanding of decimals mostly in context of money</p> <p>Level 3: The shift to the concept of fraction as a single number; a number-line representation; emergent understanding of improper fraction; early integration of fraction and decimal notation (benchmarks); fraction as a measure</p>	<p>2.MD.8. Solve word problems involving dollar bills, quarters, dimes, nickels, and pennies using \$ and ¢ symbols appropriately.</p> <p>3.NF.1. Understand a fraction <math>1/b</math> as the quantity formed by one part when a whole is partitioned into <math>b</math> equal parts; understand a fraction <math>a/b</math> as the quantity formed by <math>a</math> parts of size <math>1/b</math>.</p> <p>3.NF.3. Explain equivalence of fractions in <b>special cases</b>, and compare fractions by reasoning about their size.</p> <p>4.NF.3. Understand a fraction <math>a/b</math> with <math>a &gt; 1</math> as a sum of fractions <math>1/b</math>.</p> <p>3.NF.2. Understand a fraction as a number on the number line; represent fractions on a number line diagram.</p> <p>4.NF.1. Explain why a fraction <math>a/b</math> is equivalent to a fraction <math>(n \times a)/(n \times b)</math> by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.</p> <p>4.NF.2. Compare two fractions with different numerators and different denominators (e.g., by creating common denominators or numerators) or by comparing to a benchmark fraction such as <math>1/2</math>. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols <math>&gt;</math>, <math>=</math>, or <math>&lt;</math>, and justify the conclusions (e.g., by using a visual fraction model).</p> <p>4.NF.5. Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100. For example, express <math>3/10</math> as <math>30/100</math>, and add <math>3/10 + 4/100 = 34/100</math>.</p> <p>4.NF.6. Use decimal notation for fractions with denominators 10 or 100. For example, rewrite 0.62 as <math>62/100</math>; describe a length as 0.62 m; locate 0.62 on a number-line diagram.</p> <p>4.NF.7. Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols <math>&gt;</math>, <math>=</math>, or <math>&lt;</math>, and justify the conclusions (e.g., by using a visual model).</p> <p>5.NF.2. Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators (e.g., by using visual fraction models or equations to represent the problem). Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers. For example, recognize an incorrect result <math>2/5 + 1/2 = 3/7</math> by observing that <math>3/7 &lt; 1/2</math>.</p>

Table B1: Continued

Level in the LP	Relevant CCSS (NGA & CCSSO, 2010)
Level 4: Representational fluency including smooth translating between different notations of rational numbers; partitioning multiple units; using partitioning in flexible ways (partitioning a remainder); established multiplication of fractions with fractions (i.e., partitioning of a partitioning; partitioning a third into fourths results in 1/12 size parts); fraction as an operator	<p>4.NF.4. Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.</p> <p>5.NF.1. Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators. For example, <math>2/3 + 5/4 = 8/12 + 15/12 = 23/12</math>. (In general, <math>a/b + c/d = (ad + bc)/bd</math>.)</p> <p>5.NBT.1. Recognize that in a multidigit number, a digit in one place represents 10 times as much as it represents in the place to its right and 1/10 of what it represents in the place to its left.</p> <p>5.NBT.2. Explain patterns in the number of zeros of the product when multiplying a number by powers of 10, and explain patterns in the placement of the decimal point when a decimal is multiplied or divided by a power of 10. Use whole-number exponents to denote powers of 10.</p> <p>5.NBT.3. Read, write, and compare decimals to thousandths.</p> <ul style="list-style-type: none"> <li>• Read and write decimals to thousandths using base-ten numerals, number names, and expanded form, for example, <math>347.392 = 3 \times 100 + 4 \times 10 + 7 \times 1 + 3 \times (1/10) + 9 \times (1/100) + 2 \times (1/1000)</math>.</li> <li>• Compare two decimals to thousandths based on meanings of the digits in each place, using <math>&gt;</math>, <math>=</math>, and <math>&lt;</math> symbols to record the results of comparisons.</li> </ul> <p>5.NBT.4. Use place-value understanding to round decimals to any place.</p> <p>5.NBT.7. Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.</p> <p>5.NF.4. Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.</p> <ul style="list-style-type: none"> <li>• Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.</li> </ul> <p>5.NF.3. Interpret a fraction as division of the numerator by the denominator (<math>a/b = a \div b</math>). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers (e.g., by using visual fraction models or equations to represent the problem). For example, interpret <math>3/4</math> as the result of dividing 3 by 4, noting that <math>3/4</math> multiplied by 4 equals 3, and that when three wholes are shared equally among four people each person has a share of size <math>3/4</math>. If nine people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get? Between what two whole numbers does your answer lie?</p> <p>5.NF.4. Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.</p> <ul style="list-style-type: none"> <li>• Interpret the product <math>(a/b) \times q</math> as a parts of a partition of <math>q</math> into <math>b</math> equal parts; equivalently, as the result of a sequence of operations <math>a \times q \div b</math>. For example, use a visual fraction model to show <math>(2/3) \times 4 = 8/3</math>, and create a story context for this equation. Do the same with <math>(2/3) \times (4/5) = 8/15</math>. (In general, <math>(a/b) \times (c/d) = ac/bd</math>.)</li> </ul> <p>5.NF.5. Interpret multiplication as scaling (resizing), by</p> <ul style="list-style-type: none"> <li>• Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.</li> </ul>
Level 5: General model of fraction; multiplicative structure; contextual fluency; fraction as quotient	

Table B1: Continued

Level in the LP	Relevant CCSS (NGA & CCSSO, 2010)
	<ul style="list-style-type: none"> <li>Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence <math>a/b = (n \times a)/(n \times b)</math> to the effect of multiplying <math>a/b</math> by 1.</li> </ul> <p>5.NF.6. Solve real world problems involving multiplication of fractions and mixed numbers (e.g., by using visual fraction models or equations to represent the problem).</p> <p>5.NF.7. Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.</p> <ul style="list-style-type: none"> <li>Interpret division of a unit fraction by a nonzero whole number, and compute such quotients. For example, create a story context for <math>(1/3) \div 4</math>, and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that <math>(1/3) \div 4 = 1/12</math> because <math>(1/12) \times 4 = 1/3</math>.</li> <li>Interpret division of a whole number by a unit fraction, and compute such quotients. For example, create a story context for <math>4 \div (1/5)</math>, and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that <math>4 \div (1/5) = 20</math> because <math>20 \times (1/5) = 4</math>.</li> <li>Solve real world problems involving division of unit fractions by nonzero whole numbers and division of whole numbers by unit fractions (e.g., by using visual fraction models and equations to represent the problem). For example, how much chocolate will each person get if three people share <math>1/2</math> lb. of chocolate equally? How many <math>1/3</math>-cup servings are in 2 cups of raisins?</li> </ul>

In these cases, we list the standard at the earliest level in which it is relevant, as it could be argued that all knowledge in the progression continues to build a more in-depth understanding of even the simplest standards.

**Action Editor:** James Carlson

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